Published by Institute of Physics Publishing for SISSA

Received: *August 6, 2005*
Revised: *January 25, 2006*
Accepted: *January 31, 2006*
Published: *February 15, 2006*

On five-dimensional superspaces

Sergei M. Kuzenko

School of Physics M013, The University of Western Australia
35 Stirling Highway, Crawley W.A. 6009, Australia
E-mail: kuzenko@cyllene.uwa.edu.au

William D. Linch III[*]

Center for String and Particle Theory, Department of Physics, University of Maryland
College Park, MD 20742-4111 U.S.A.
E-mail: ldw@physics.umd.edu

Abstract: Recent one-loop calculations of certain supergravity-mediated quantum corrections in supersymmetric brane-world models employ either the component formulation (hep-th/0305184) or the superfield formalism with only half of the bulk supersymmetry manifestly realized (hep-th/0305169 and hep-th/0411216). There are reasons to expect, however, that 5D supergraphs provide a more efficient setup to deal with these and more involved (in particular, higher-loop) calculations. As a first step toward elaborating such supergraph techniques, we develop in this letter a manifestly supersymmetric formulation for 5D globally supersymmetric theories with eight supercharges. Simple rules are given to reduce 5D superspace actions to a hybrid form which keeps manifest only the 4D, $\mathcal{N} = 1$ Poincaré supersymmetry. (Previously, such hybrid actions were carefully worked out by rewriting the component actions in terms of simple superfields). To demonstrate the power of this formalism for model building applications, two families of off-shell supersymmetric nonlinear sigma-models in five dimensions are presented (including those with cotangent bundles of Kähler manifolds as target spaces). We elaborate, trying to make our presentation maximally clear and self-contained, on the techniques of 5D harmonic and projective superspaces used at some stages in this letter.

Keywords: Superspaces, Field Theories in Higher Dimensions, Supersymmetry Phenomenology, Extended Supersymmetry.

[*]Address after September 1, 2005: C. N. Yang Institute for Theoretical Physics and the Department of Mathematics, State University of New York, Stony Brook, NY 11794, U.S.A.

JHEP02(2006)038

Contents

1. Introduction **2**

2. 5D supersymmetric matter **6**
- 2.1 Vector multiplet in conventional superspace 6
- 2.2 Vector multiplet in harmonic superspace 7
- 2.3 Vector multiplet in components 10
- 2.4 Fayet-Sohnius hypermultiplet 10
- 2.5 Off-shell hypermultiplets without central charge 11

3. Supersymmetric actions **12**
- 3.1 Four-derivative vector multiplet action 13
- 3.2 Multiplets with intrinsic central charge 13
- 3.3 Fayet-Sohnius hypermultiplet 14
- 3.4 Vector multiplet 14

4. Chern-Simons couplings **15**

5. 5D supermultiplets in reduced superspace **17**
- 5.1 Vector multiplet 17
- 5.2 Fayet-Sohnius hypermultiplet 19

6. Projective superspace and dimensional reduction **19**
- 6.1 Doubly punctured harmonic superspace 19
- 6.2 Tensor multiplet and nonlinear sigma-models 22
- 6.3 Polar hypermultiplet and nonlinear sigma-models 23

7. Vector multiplet in projective superspace **25**

8. Conclusion **29**

A. 5D notation and conventions **29**

B. Tensor fields on the two-sphere **32**

C. Projective superspace action **33**

1. Introduction

Supersymmetric field theories in dimensions higher than the four accessible in our everyday experiences have been contemplated for many years now. Besides being forced on us by our current understanding of superstring theory, it has also proven to be of possible phenomenological importance in which discipline such theories are known as supersymmetric "brane-world" models [1 − 4]. Currently, this type of theory is being used by various groups in attempts to implement supersymmetry breaking in a manner consistent with the stringent bounds coming from flavor changing neutral currents. One particular application uses a supersymmetric gravitational theory in a five-dimensional spacetime of which the "extra" spacial dimension is a compact interval of some length[1] ℓ [4, 5]. At each end of the

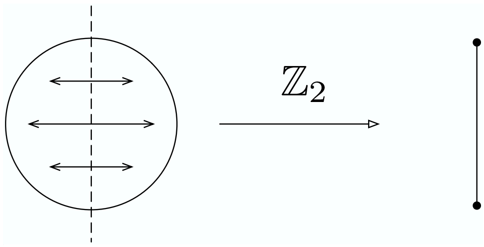

Figure 1: The orbifold construction.

interval is a copy of 4D Minkowski spacetime (the branes). Brane world models consist of postulating that the standard model of particle physics is localized on one of these branes (the "visible" or "infra-red" brane) while other fields propagate in the interior ("bulk") of the 5D spacetime or on the other "hidden" (or "ultraviolet") brane (see figure 2). In the particular model under consideration, supersymmetry is broken on the hidden brane by allowing the F- (or D-term [6]) of some field localized on that brane to acquire a vacuum expectation value. This is then communicated to our brane by super-graviton loops in the bulk which induce the breakdown of supersymmetry on the standard model brane. The final result is a four-dimensional effective action for the standard model fields on the visible brane.

Needless to say, calculations such as that of gravitational loop corrections are difficult to perform in components (although it was done in [7] at one loop). On the other hand, it is commonly believed that the development of a full-fledged 5D superspace formulation has the annoying drawback that the result, which is desired to be a four-dimensional effective action, is given in a complicated form. More exactly, the output of such an effort is manifestly supersymmetric in five dimensions and must be dimensionally reduced in the final

[1] An elegant way of constructing such a space is to start with a 5D Minkowski space and toroidally compactify one of the spacial directions, producing a space of topology $\mathbb{R}^4 \times S^1$. One subsequently defines a non-free action of $\mathbb{Z}_2$ on the circle which has two antipodal fixed points (i.e. a reflection through a "diameter") and mods the circle by this action. Since the action was not free, the quotient $S^1/\mathbb{Z}_2$ is not a smooth manifold (figure 1). Nevertheless it is a manifold-with-boundary diffeomorphic to the closed finite interval $[0, 1]$. The "orbifold" $\mathbb{R}^4 \times (S^1/\mathbb{Z}_2)$ has as boundary two hyperplanes (the "orbifold fixed planes"), each isometric to a 4D Minkowski space, at the fixed points of the $\mathbb{Z}_2$ action.

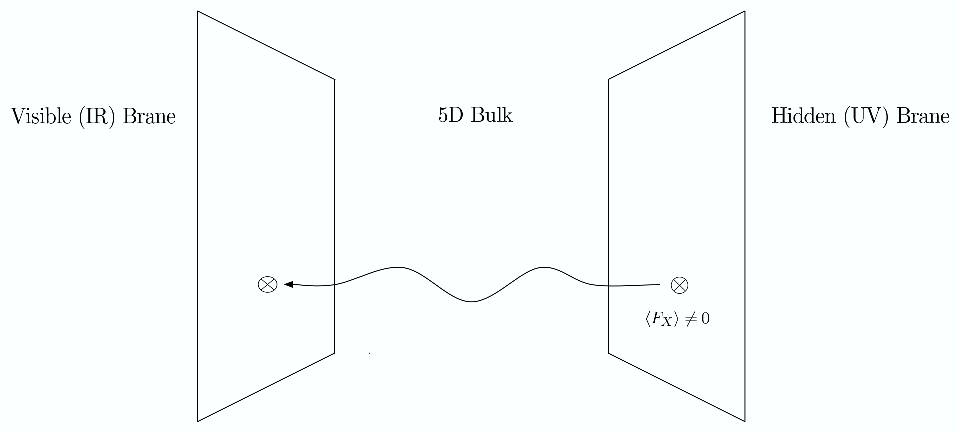

Figure 2: The 5D brane-world scenario of the gravitational mediation of supersymmetry breaking in the standard model. The F-term of a chiral field X on the hidden brane acquires a vacuum excitation value. This is then communicated to the standard model (visible) brane by gravitational messengers propagating through the bulk.

stages of the calculation. For this reason a "hybrid" formalism was developed for supergravity in five dimensions which keeps manifest only 4D, $\mathcal{N} = 1$ super-Poincaré invariance [8]. This hybrid is given in terms of supergravity prepotentials which allows one to apply the powerful supergraph techniques necessary for perturbative quantum calculations. Indeed, in [9] it was used to compute, in a more economical way than in the component approach of [7], the leading gravity loop contribution to supersymmetry breakdown described above in a very simple way.

Although the formalism was successfully extended to allow a "warping" of the extra dimension and the gravity-mediation scenario investigated in this background [10], it has a major drawback which arises as follows. This approach is essentially a superfield Noether procedure in which one starts with a linearized supergravity action, and then tries to reconstruct interaction terms, order by order, by consistently deforming the gauge transformations, etc. Usually the Noether procedure can be completed if it requires a finite number of iterations, as is the case with polynomial actions. But superfield supergravity is a highly nonlinear theory in terms of its prepotentials (see [11, 12] for reviews). As a result, the limitations of this hybrid approach are called into question. More importantly, it turns out to be difficult to discover the rules governing the coupling of this theory to other matter fields in the bulk. In the end, we are forced to turn to the known (full-fledged) off-shell formulations for 5D simple supergravity,[2] with or without supersymmetric matter, in the hope of deducing a useful superfield formulation.

Off-shell 5D simple supergravity was sketched in superspace, a quarter century ago, by Breitenlohner and Kabelschacht [13] and Howe [14] (building on a related work [15]). More recently, it was carefully elaborated by Zucker [16] at the component level, and finally perfected in [17, 18] within the superconformal tensor calculus.

[2]We prefer to use the term "5D simple supergravity," since in the literature 5D simple supersymmetry is called sometimes $\mathcal{N} = 1$ and sometimes $\mathcal{N} = 2$, depending upon taste and background.

Using the results of the 5D superconformal tensor calculus for supergravity-matter systems, one can develop a hybrid $\mathcal{N}=1$ superspace formalism by fitting the component multiplets into superfields. Such a program has been carried out in [19]. Although useful for tree-level phenomenological applications, we believe this approach is not the optimum (economical) formulation for doing supergraph loop calculations. The point is that the superconformal tensor calculus usually corresponds to a Wess-Zumino gauge in superfield supergravity. But such gauge conditions are impractical as far as supergraph calculations are concerned.

When comparing the superconformal tensor calculus for 5D simple supergravity [17, 18] with that for 4D, $\mathcal{N}=2$ and 6D, $\mathcal{N}=(1,0)$ supergravities (see [17, 18] for the relevant references), it is simply staggering how similar these formulations are, modulo some fine details. From the point of view of a superspace practitioner, the reason for this similarity is that the three versions of superconformal calculus are generated from (correspond to a Wess-Zumino gauge for) a harmonic superspace formulation for the corresponding supergravity theory, and such harmonic superspace formulations[3] look almost identical in the space-time dimensions 4, 5 and 6, again modulo fine details. For example, independent of the space-time dimension, the Yang-Mills supermultiplet is described by (formally) the same gauge superfield $\mathcal{V}^{++}$, with the same gauge freedom $\delta\mathcal{V}^{++}=-\mathcal{D}^{++}\lambda$, and with the same Wess-Zumino-type gauge

$$\mathrm{i}\,\mathcal{V}^{++}=\theta^+\Gamma^m\theta^+A_m(x)+\theta^+\Gamma^5\theta^+A_5(x)+\theta^+\Gamma^6\theta^+A_6(x)\;+\;O(\theta^3)\,,$$

where θ^+ is a four-component anticommuting spinor variable, and $m=0,1,2,3$.

The concept of harmonic superspace was originally developed for 4D, $\mathcal{N}=2$ supersymmetric theories including supergravity [20], and by now it has become a textbook subject[4] [21]. Actually it can be argued that harmonic superspace is a natural framework for all supersymmetric theories with eight supercharges, both at the classical and quantum levels. In the case of four space-time dimensions, probably the main objection to this approach was the issue that theories in harmonic superspace are often difficult to reduce to $\mathcal{N}=1$ superfields (the kind of reduction which brane-world practitioners often need). But this objection has been lifted since the advent and subsequent perfection of 4D, $\mathcal{N}=2$ projective superspace [23, 24] which allows a nice reduction to $\mathcal{N}=1$ superfields and which appears to be a truncated version of the harmonic superspace [25].

What is the difference between harmonic superspace and projective superspace? In five space-time dimensions (to be concrete), they make use of the same supermanifold $\mathbb{R}^{5|8}\times S^2$, with $\mathbb{R}^{5|8}$ the conventional 5D simple superspace. In harmonic superspace, one deals with so-called *Grassmann analytic* (also known as *twisted chiral*) superfields that are chosen to be smooth tensor fields on S^2. In projective superspace, one also deals with Grassmann analytic superfields that are holomorphic functions on an open subset of S^2. It is clear that the harmonic superspace setting is more general. Actually, many results originally

[3]The harmonic superspace formulation for 4D, $\mathcal{N}=2$ supergravity is reviewed in the book [21]. For the case of 6D, $\mathcal{N}=(1,0)$ supergravity, such a formulation was constructed in [22], and it can be used to derive a relevant formulation for 5D simple supergravity by dimensional reduction.

[4]The book [21] contains a list of relevant publications in the context of harmonic superspace.

JHEP02(2006)038

obtained in projective superspace can be reproduced from harmonic superspace by applying special truncation procedures [25]. The remarkable features of projective superspace are that (i) the projective supermultiplets are easily represented as a direct sum of standard 4D, $\mathcal{N} = 1$ superfields; (ii) this approach provides simple rules to construct low-energy effective actions that are easily expressed in terms of 4D, $\mathcal{N} = 1$ superfields. Of course, one could wonder why both harmonic and projective superspaces should be introduced? The answer is that, in many respects, they are complementary to each other. (This is analogous to the relation between the theorems of existence of solutions for differential equations and concrete techniques to solve such equations.)

To avoid technicalities, in this paper we do not consider 5D superfield supergravity at all, and concentrate only on developing a 5D simple superspace approach to globally supersymmetric gauge theories. One of our main objectives is to demonstrate that 5D superspace may be useful, even in the context of 4D effective theories with an extra dimension. Here we develop manifestly 5D supersymmetric techniques which, on the one hand, allow us to construct many of the 5D supersymmetric models originally developed within the "hybrid" formulation. One the other hand, these techniques make it possible to construct very interesting supersymmetric nonlinear sigma-models whose construction is practically beyond the scope of the "hybrid" formulation. Examples of such 5D supersymmetric sigma-models are constructed for the first time below. We therefore believe that the paper should be of some interest to both superspace experts and newcomers.

It is worth saying a few words about the global structure of this paper. We are aiming at (i) elaborating 5D off-shell matter supermultiplets and their superfield descriptions; (ii) developing various universal procedures to construct manifestly 5D supersymmetric action functionals, and then applying them to specific supermultiplets; (iii) elaborating on techniques to reduce such super-actions to 4D, $\mathcal{N} = 1$ superfields. New elements of 5D superfield formalism are introduced only if they are essential for further consideration. For example, the Yang-Mills off-shell supermultiplet can be realized in 5D conventional superspace in terms of constrained superfields. In order to solve the constraints, however, one has to introduce the concept of harmonic superspace.

This paper is organized as follows: In section 2 we describe, building on earlier work [15, 26], the 5D Yang-Mills supermultiplet and its salient properties, both in the conventional and harmonic superspaces. We also describe several off-shell realizations for the 5D hypermultiplet. In section 3 we present two procedures to construct 5D manifestly supersymmetric actions for multiplets with and without intrinsic central charge, and give several examples. Section 4 is devoted to 5D supersymmetric Chern-Simons theories. Their harmonic superspace actions are given in a new form, as compared with [26], which allows a simple reduction to the projective superspace. We also uncover the 5D origin for the superfield constraints describing the so-called 4D, $\mathcal{N} = 2$ nonlinear vector-tensor multplet. In section 5, some of the results developed in the previous sections are reduced to a "hybrid" formulation which keeps manifest only 4D, $\mathcal{N} = 1$ super Poincaré symmetry. Section 6 introduces 5D simple projective superspace and projective multiplets. Here we also present two families of 5D off-shell supersymmetric nonlinear sigma-models which are formulated, respectively, in terms of a (i) 5D tensor multiplet; (ii) 5D polar mutiplet. Sec-

JHEP02(2006)038

tion 7 deals with the vector multiplet in projective superspace. A brief conclusion is given in section 8. This paper also includes three technical appendices. Appendix A contains our 5D notation and conventions, inspired by those in [27, 28], as well as some important identities. Appendix B is devoted to a review of the well-known one-to-one correspondence between smooth tensor fields on $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ and smooth scalar functions over SU(2) with definite U(1) charges. Finally, in appendix C we briefly demonstrate, mainly following [25], how to derive the projective superspace action (6.14) from the harmonic superspace action (3.2).

2. 5D supersymmetric matter

2.1 Vector multiplet in conventional superspace

To describe a Yang-Mills supermultiplet in 5D simple superspace $\mathbb{R}^{5|8}$ parametrized by coordinates $z^{\hat{A}} = (x^{\hat{a}}, \theta^{\hat{\alpha}}_i)$ we introduce gauge-covariant derivatives[5]

$$\mathcal{D}_{\hat{A}} = (\mathcal{D}_{\hat{a}}, \mathcal{D}^i_{\hat{\alpha}}) = D_{\hat{A}} + \mathrm{i}\,\mathcal{V}_{\hat{A}}(z)\,, \qquad [\mathcal{D}_{\hat{A}}\,, \mathcal{D}_{\hat{B}}\} = T_{\hat{A}\hat{B}}{}^{\hat{C}}\,\mathcal{D}_{\hat{C}} + C_{\hat{A}\hat{B}}\,\Delta + \mathrm{i}\,\mathcal{F}_{\hat{A}\hat{B}}\,, \tag{2.1}$$

with $D_{\hat{A}} = (\partial_{\hat{a}}, D^i_{\hat{\alpha}})$ the flat covariant derivatives obeying the anti-commutation relations (A.15), Δ the central charge, and $\mathcal{V}_{\hat{A}}$ the gauge connection taking its values in the Lie algebra of the gauge group. The connection is chosen to be inert under the central charge transformations, $[\Delta\,, \mathcal{V}_{\hat{A}}] = 0$. The operators $\mathcal{D}_{\hat{A}}$ possess the following gauge transformation law

$$\mathcal{D}_{\hat{A}} \;\mapsto\; \mathrm{e}^{\mathrm{i}\tau(z)}\,\mathcal{D}_{\hat{A}}\,\mathrm{e}^{-\mathrm{i}\tau(z)}\,, \qquad \tau^\dagger = \tau\,, \qquad [\Delta\,, \tau] = 0\,, \tag{2.2}$$

with the gauge parameter $\tau(z)$ being arbitrary modulo the reality condition imposed. The gauge-covariant derivatives are required to obey some constraints [15] such that

$$\begin{aligned}
\{\mathcal{D}^i_{\hat{\alpha}}\,, \mathcal{D}^j_{\hat{\beta}}\} &= -2\mathrm{i}\,\varepsilon^{ij}\,\Big((\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\,\mathcal{D}_{\hat{c}} + \varepsilon_{\hat{\alpha}\hat{\beta}}\,(\Delta + \mathrm{i}\,\mathcal{W})\Big)\,, \qquad [\mathcal{D}^i_{\hat{\alpha}}\,, \Delta] = 0\,, \\
\big[\mathcal{D}_{\hat{a}}\,, \mathcal{D}^j_{\hat{\beta}}\big] &= \mathrm{i}\,(\Gamma_{\hat{a}})_{\hat{\beta}}{}^{\hat{\gamma}}\,\mathcal{D}^j_{\hat{\gamma}}\mathcal{W}\,, \qquad \big[\mathcal{D}_{\hat{a}}\,, \mathcal{D}_{\hat{b}}\big] = -\frac{1}{4}(\Sigma_{\hat{a}\hat{b}})^{\hat{\alpha}\hat{\beta}}\,\mathcal{D}^i_{\hat{\alpha}}\mathcal{D}_{\hat{\beta}i}\mathcal{W} = \mathrm{i}\,\mathcal{F}_{\hat{a}\hat{b}}\,,
\end{aligned} \tag{2.3}$$

with the matrices $\Gamma_{\hat{a}}$ and $\Sigma_{\hat{a}\hat{b}}$ defined in appendix A. Here the field strength $\mathcal{W}$ is hermitian, $\mathcal{W}^\dagger = \mathcal{W}$, and obeys the Bianchi identity (see e.g. [26])

$$\mathcal{D}^{(i}_{\hat{\alpha}}\mathcal{D}^{j)}_{\hat{\beta}}\mathcal{W} = \frac{1}{4}\varepsilon_{\hat{\alpha}\hat{\beta}}\,\mathcal{D}^{\hat{\gamma}(i}\mathcal{D}^{j)}_{\hat{\gamma}}\mathcal{W}\,, \tag{2.4}$$

and therefore

$$\mathcal{D}^{(i}_{\hat{\alpha}}\mathcal{D}^{j}_{\hat{\beta}}\mathcal{D}^{k)}_{\hat{\gamma}}\mathcal{W} = 0\,. \tag{2.5}$$

The independent component fields contained in $\mathcal{W}$ are:

$$\varphi = \mathcal{W}\|\,, \quad \mathrm{i}\,\Psi^i_{\hat{\alpha}} = \mathcal{D}^i_{\hat{\alpha}}\mathcal{W}\|\,, \quad -4\mathrm{i}\,F_{\hat{\alpha}\hat{\beta}} = \mathcal{D}^i_{(\hat{\alpha}}\mathcal{D}_{\hat{\beta})i}\mathcal{W}\|\,, \quad -4\mathrm{i}\,X^{ij} = \mathcal{D}^{\hat{\alpha}(i}\mathcal{D}^{j)}_{\hat{\alpha}}\mathcal{W}\|\,. \tag{2.6}$$

Here and in what follows, $U\|$ denotes the θ-independent component of a superfield $U(x,\theta)$. It is worth noting that

$$F_{\hat{a}\hat{b}} = \mathcal{F}_{\hat{a}\hat{b}}\|\,. \tag{2.7}$$

[5] Our notation and conventions are collected in appendix A.

2.2 Vector multiplet in harmonic superspace

The most elegant way to solve the constraints encoded in the algebra (2.3) is to use the concept of harmonic superspace originally developed for 4D, $\mathcal{N} = 2$ supersymmetric theories [20, 21] (related ideas appeared in [29]). In this approach, the conventional superspace $\mathbb{R}^{5|8}$ is embedded into $\mathbb{R}^{5|8} \times S^2$, where the two-sphere $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ is parametrized by so-called harmonic $u_i{}^-$ and $u_i{}^+$, that is group elements

$$(u_i{}^- \,, u_i{}^+) \in \mathrm{SU}(2)\,, \quad u_i^+ = \varepsilon_{ij}\, u^{+j}\,, \quad (u^{+i})^* = u_i^-\,, \quad u^{+i}u_i^- = 1\,. \tag{2.8}$$

As is well-known, tensor fields over S^2 are in a one-to-one correspondence with functions over SU(2) possessing definite harmonic U(1) charge (see [25] for a review). A function $\Psi^{(p)}(u)$ is said to have harmonic U(1) charge p if

$$\Psi^{(p)}(\mathrm{e}^{\mathrm{i}\alpha}u^+, \mathrm{e}^{-\mathrm{i}\alpha}u^-) = \mathrm{e}^{\mathrm{i}p\alpha}\, \Psi^{(p)}(u^+, u^-)\,, \qquad |\mathrm{e}^{\mathrm{i}\alpha}| = 1\,. \tag{2.9}$$

Such functions, extended to the whole harmonic superspace $\mathbb{R}^{5|8} \times S^2$, that is $\Psi^{(p)}(z, u)$, are called harmonic superfields. Introducing the harmonic derivatives [20]

$$\begin{gathered} D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}}\,, \quad D^{--} = u^{-i}\frac{\partial}{\partial u^{+i}}\,, \quad D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,, \\ [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,, \qquad [D^{++}, D^{--}] = D^0\,, \end{gathered} \tag{2.10}$$

one can see that D^0 is the operator of harmonic U(1) charge, $D^0\,\Psi^{(p)}(z, u) = p\,\Psi^{(p)}(z, u)$. Defining

$$\mathcal{D}_{\mathbf{A}} \equiv (\mathcal{D}_{\hat{A}}, \mathcal{D}^{++}, \mathcal{D}^{--}, \mathcal{D}^0)\,, \qquad \mathcal{D}^{\pm\pm} = D^{\pm\pm}\,, \qquad \mathcal{D}^0 = D^0\,, \tag{2.11}$$

one observes that the operators $\mathcal{D}_{\mathbf{A}}$ possess the same transformation law (2.2) as $\mathcal{D}_{\hat{A}}$.

Introduce a new basis for the spinor covariant derivatives: $\mathcal{D}^+_{\hat\alpha} = \mathcal{D}^i_{\hat\alpha}\, u_i^+$ and $\mathcal{D}^-_{\hat\alpha} = \mathcal{D}^i_{\hat\alpha}\, u_i^-$. Then, eq. (2.3) leads to

$$\begin{aligned} \{\mathcal{D}^+_{\hat\alpha}\,, \mathcal{D}^+_{\hat\beta}\} &= 0\,, & [\mathcal{D}^{++}\,, \mathcal{D}^+_{\hat\alpha}] &= 0\,, \\ \{\mathcal{D}^+_{\hat\alpha}\,, \mathcal{D}^-_{\hat\beta}\} &= 2\mathrm{i}\left(\mathcal{D}_{\hat\alpha\hat\beta} + \varepsilon_{\hat\alpha\hat\beta}\,(\Delta + \mathrm{i}\,\mathcal{W})\right)\,, && \\ [\mathcal{D}^{++}\,, \mathcal{D}^-_{\hat\alpha}] &= \mathcal{D}^+_{\hat\alpha}\,, & [\mathcal{D}^{--}\,, \mathcal{D}^+_{\hat\alpha}] &= \mathcal{D}^-_{\hat\alpha}\,. \end{aligned} \tag{2.12}$$

In harmonic superspace, the integrability condition $\{\mathcal{D}^+_{\hat\alpha}\,, \mathcal{D}^+_{\hat\beta}\} = 0$ is solved by

$$\mathcal{D}^+_{\hat\alpha} = \mathrm{e}^{-\mathrm{i}\Omega}\, D^+_{\hat\alpha}\, \mathrm{e}^{\mathrm{i}\Omega}\,, \tag{2.13}$$

for some Lie-algebra valued harmonic superfield $\Omega = \Omega(z, u)$ of vanishing harmonic U(1) charge, $D^0\,\Omega = 0$. This superfield is called the bridge. The bridge possesses a richer gauge freedom than the original τ-group (2.2)

$$\mathrm{e}^{\mathrm{i}\Omega(z,u)} \;\mapsto\; \mathrm{e}^{\mathrm{i}\lambda(z,u)}\, \mathrm{e}^{\mathrm{i}\Omega(z,u)}\, \mathrm{e}^{-\mathrm{i}\tau(z)}\,, \qquad D^+_{\hat\alpha}\,\lambda = 0\,, \qquad [\Delta\,, \lambda] = 0\,. \tag{2.14}$$

The λ- and τ-transformations generate, respectively, the so-called λ- and τ-groups.

JHEP02(2006)038

One can now define *covariantly analytic* superfields constrained by

$$\mathcal{D}^+_{\hat\alpha}\Phi^{(p)} = 0 \ . \tag{2.15}$$

Here $\Phi^{(p)}(z,u)$ carries U(1)-charge p, $D^0\Phi^{(p)} = p\Phi^{(p)}$, and can be represented as follows

$$\Phi^{(p)} = \mathrm{e}^{-\mathrm{i}\Omega}\phi^{(p)} \,, \qquad D^+_{\hat\alpha}\phi^{(p)} = 0 \,, \tag{2.16}$$

with $\phi^{(p)}(\zeta)$ being an *analytic* superfield — that is, a field over the so-called *analytic subspace* of the harmonic superspace parametrized by

$$\zeta \equiv \{\boldsymbol{x}^{\hat a}, \theta^{+\hat\alpha}, u^+_i, u^-_j\} \,, \tag{2.17}$$

where

$$\boldsymbol{x}^{\hat a} = x^{\hat a} + \mathrm{i}\,(\Gamma^{\hat a})_{\hat\beta\hat\gamma}\,\theta^{+\hat\beta}\theta^{-\hat\gamma} \,, \qquad \theta^\pm_{\hat\alpha} = \theta^i_{\hat\alpha}\,u^\pm_i \ . \tag{2.18}$$

In particular, the gauge parameter λ in (2.14) is an unconstrained analytic superfield of vanishing harmonic U(1) charge, $D^0\lambda = 0$. It is clear that the superfields $\Phi^{(p)}$ and $\phi^{(p)}$ describe the same matter multiplet but in different frames (or, equivalently, representations), and they transform under the τ- and λ-gauge groups, respectively.

$$\Phi^{(p)}(z,u) \ \mapsto \ \mathrm{e}^{\mathrm{i}\tau(z)}\,\Phi^{(p)}(z,u) \,, \qquad \phi^{(p)}(z,u) \ \mapsto \ \mathrm{e}^{\mathrm{i}\lambda(z,u)}\,\phi^{(p)}(z,u) \ . \tag{2.19}$$

By construction, the analytic subspace (2.17) is closed under the supersymmetry transformations. Unlike the chiral subspace, it is real with respect to the generalized conjugation (often called the smile-conjugation) $\breve{}$ [20] defined to be the composition of the complex conjugation (Hermitian conjugation in the case of Lie-algebra-valued superfields) with the operation * acting on the harmonics only

$$(u^+_i)^\star = u^-_i \,, \qquad (u^-_i)^\star = -u^+_i \quad \Rightarrow \quad (u^\pm_i)^{\star\star} = -u^\pm_i \,, \tag{2.20}$$

hence

$$(u^{+i})\breve{} = -u^+_i \qquad\qquad (u^-_i)\breve{} = u^{-i} \ . \tag{2.21}$$

The analytic superfields of even U(1) charge may therefore be chosen to be real. In particular, the bridge Ω and the gauge parameter λ are real.

The covariant derivatives in the λ-frame are obtained from those in the τ-frame, eq. (2.11), by applying the transformation

$$\mathcal{D}_{\mathbf{A}} \quad \mapsto \quad \mathrm{e}^{\mathrm{i}\Omega}\,\mathcal{D}_{\mathbf{A}}\,\mathrm{e}^{-\mathrm{i}\Omega} \ . \tag{2.22}$$

Then, the gauge transformation of the covariant derivatives becomes

$$\mathcal{D}_{\mathbf{A}} \ \mapsto \ \mathrm{e}^{\mathrm{i}\lambda(\zeta)}\,\mathcal{D}_{\mathbf{A}}\,\mathrm{e}^{-\mathrm{i}\lambda(\zeta)} \,, \qquad \breve\lambda = \lambda \,, \qquad [\Delta\,,\lambda] = 0 \ . \tag{2.23}$$

In the λ-frame, the spinor covariant derivatives $\mathcal{D}^+_{\hat\alpha}$ coincide with the flat ones, $\mathcal{D}^+_{\hat\alpha} = D^+_{\hat\alpha}$, while the harmonic covariant derivatives acquire connections,

$$\mathcal{D}^{\pm\pm} = D^{\pm\pm} + \mathrm{i}\,\mathcal{V}^{\pm\pm} \ . \tag{2.24}$$

The *real* connection $\mathcal{V}^{++}$ is seen to be an analytic superfield, $D^+_{\hat{\alpha}}\mathcal{V}^{++} = 0$, of harmonic U(1) charge plus two, $D^0\,\mathcal{V}^{++} = 2\mathcal{V}^{++}$. The other harmonic connection $\mathcal{V}^{--}$ turns out to be uniquely determined in terms of $\mathcal{V}^{++}$ using the zero-curvature condition

$$[\mathcal{D}^{++}\,,\mathcal{D}^{--}] = D^0 \quad \Longleftrightarrow \quad D^{++}\mathcal{V}^{--} - D^{--}\mathcal{V}^{++} + \mathrm{i}\,[\mathcal{V}^{++}\,,\mathcal{V}^{--}] = 0\,, \tag{2.25}$$

as demonstrated in [30]. The result is

$$\mathcal{V}^{--}(z,u) = \sum_{n=1}^{\infty}(-\mathrm{i})^{n+1}\int \mathrm{d}u_1\ldots\mathrm{d}u_n\,\frac{\mathcal{V}^{++}(z,u_1)\,\mathcal{V}^{++}(z,u_2)\cdots\mathcal{V}^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\ldots(u_n^+u^+)}\,, \tag{2.26}$$

with $(u_1^+u_2^+) = u_1^{+i}u^+_{2\,i}$, and the harmonic distributions on the right of (2.26) defined, e.g., in [21]. Integration over the group manifold SU(2) is normalized according to [20]

$$\int \mathrm{d}u\,1 = 1\,, \qquad \int \mathrm{d}u\,u^+_{(i_1}\cdots u^+_{i_n}\,u^-_{j_1}\cdots u^-_{j_m)} = 0\,, \quad n+m>0\,. \tag{2.27}$$

As far as the connections $\mathcal{V}^-_{\hat{\alpha}}$ and $\mathcal{V}_{\hat{a}}$ are concerned, they can be expressed in terms of $\mathcal{V}^{--}$ with the aid of the (anti-)commutation relations

$$\left[\mathcal{D}^{--}\,,\mathcal{D}^+_{\hat{\alpha}}\right] = \mathcal{D}^-_{\hat{\alpha}}\,, \qquad \{\mathcal{D}^+_{\hat{\alpha}}\,,\mathcal{D}^-_{\hat{\beta}}\} = 2\mathrm{i}\left(\mathcal{D}_{\hat{\alpha}\hat{\beta}} + \varepsilon_{\hat{\alpha}\hat{\beta}}\,(\Delta + \mathrm{i}\,\mathcal{W}_\lambda)\right)\,. \tag{2.28}$$

In particular, one obtains

$$\mathcal{W}_\lambda = \frac{\mathrm{i}}{8}(\hat{D}^+)^2\,\mathcal{V}^{--}\,, \qquad (\hat{D}^+)^2 = D^{+\hat{\alpha}}D^+_{\hat{\alpha}}\,, \tag{2.29}$$

where $\mathcal{W}_\lambda$ stands for the field strength in the λ-frame. Therefore, $\mathcal{V}^{++}$ is the single unconstrained analytic prepotential of the theory. With the aid of (2.25) one can obtain the following useful expression

$$\mathcal{W} = \frac{\mathrm{i}}{8}\int \mathrm{d}u\,(\hat{D}^-)^2\,\mathcal{V}^{++} \;+\; O\Big((\mathcal{V}^{++})^2\Big)\,. \tag{2.30}$$

In the Abelian case, only the first term on the right survives. In what follows, we do not distinguish between $\mathcal{W}$ and $\mathcal{W}_\lambda$.

With the notation $(\hat{\mathcal{D}}^+)^2 = \mathcal{D}^{+\hat{\alpha}}\mathcal{D}^+_{\hat{\alpha}}$, the Bianchi identity (2.4) takes the form

$$\mathcal{D}^+_{\hat{\alpha}}\mathcal{D}^+_{\hat{\beta}}\mathcal{W} = \frac{1}{4}\varepsilon_{\hat{\alpha}\hat{\beta}}\,(\hat{\mathcal{D}}^+)^2\mathcal{W} \quad \Rightarrow \quad \mathcal{D}^+_{\hat{\alpha}}\mathcal{D}^+_{\hat{\beta}}\mathcal{D}^+_{\hat{\gamma}}\mathcal{W} = 0\,. \tag{2.31}$$

Using the Bianchi identity (2.31), one can readily construct a covariantly analytic descendant of $\mathcal{W}$

$$-\mathrm{i}\,\mathcal{G}^{++} = \mathcal{D}^{+\hat{\alpha}}\mathcal{W}\,\mathcal{D}^+_{\hat{\alpha}}\mathcal{W} + \frac{1}{4}\{\mathcal{W}\,,(\hat{\mathcal{D}}^+)^2\mathcal{W}\}\,, \qquad \mathcal{D}^+_{\hat{\alpha}}\mathcal{G}^{++} = \mathcal{D}^{++}\mathcal{G}^{++} = 0\,. \tag{2.32}$$

JHEP02(2006)038

2.3 Vector multiplet in components

The gauge freedom

$$-\delta\mathcal{V}^{++} = \mathcal{D}^{++}\lambda = D^{++}\lambda + \mathrm{i}\,[\mathcal{V}^{++}\,,\lambda] \tag{2.33}$$

can be used to choose a Wess-Zumino gauge of the form

$$\begin{aligned}\mathcal{V}^{++}(\boldsymbol{x},\theta^+,u) &= \mathrm{i}\,(\hat\theta^+)^2\,\varphi(\boldsymbol{x}) - \mathrm{i}\,\theta^+\Gamma^{\hat m}\theta^+\,A_{\hat m}(\boldsymbol{x}) + 4(\hat\theta^+)^2\theta^{+\hat\alpha}\,u^-_i\,\Psi^i_{\hat\alpha}(\boldsymbol{x})\\ &\quad -\frac{3}{2}\,(\hat\theta^+)^2(\hat\theta^+)^2\,u^-_i u^-_j\,X^{ij}(\boldsymbol{x})\,,\end{aligned} \tag{2.34}$$

where

$$(\hat\theta^+)^2 = \theta^{+\hat\alpha}\theta^+_{\hat\alpha}\,, \qquad \theta^+\Gamma^{\hat m}\theta^+ = \theta^{+\hat\alpha}\,(\Gamma^{\hat m})_{\hat\alpha}{}^{\hat\beta}\,\theta^+{}_{\hat\beta} = -(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,\theta^{+\hat\alpha}\theta^{+\hat\beta}\,. \tag{2.35}$$

In this gauge, the expression (2.26) simplifies considerably

$$\begin{aligned}\mathcal{V}^{--}(z,u) &= \int \mathrm{d}u_1\,\frac{\mathcal{V}^{++}(z,u_1)}{(u^+u^+_1)^2} + \frac{\mathrm{i}}{2}\int \mathrm{d}u_1\,\mathrm{d}u_2\,\frac{[\mathcal{V}^{++}(z,u_1)\,,\mathcal{V}^{++}(z,u_2)]}{(u^+u^+_1)(u^+_1u^+_2)(u^+_2u^+)}\\ &\quad +\text{terms of third- and fourth-order in } \mathcal{V}^{++}\,.\end{aligned} \tag{2.36}$$

Here the explicit form of the cubic and quartic terms is not relevant for our consideration. One of the important properties of the Wess-Zumino gauge is

$$\mathcal{D}_{\hat m}\| \equiv \partial_{\hat m} + \mathrm{i}\,\mathcal{V}_{\hat m}\| = \partial_{\hat m} + \mathrm{i}\,A_{\hat m}(x)\,. \tag{2.37}$$

The component fields of $\mathcal{W}$ and $\mathcal{V}^{++}$ can be related to each other using the identity

$$F^+_2 = (u^+_1u^+_2)\,F^-_1 - (u^-_1u^+_2)\,F^+_1\,, \qquad F^\pm = F^i\,u^\pm_i\,, \tag{2.38}$$

and the analyticity of $\mathcal{V}^{++}$. (The latter property implies, for instance, $D^+\mathcal{V}^{++}(z,u_1) = -(u^+u^+_1)\,D^-_1\mathcal{V}^{++}(z,u_1)$.) Thus one gets

$$\begin{aligned}\mathcal{W}\| &= \frac{\mathrm{i}}{8}(\hat D^+)^2\,\mathcal{V}^{--}\| = \frac{\mathrm{i}}{8}\int \mathrm{d}u_1\,(\hat D^-_1)^2\,\mathcal{V}^{++}(z,u_1)\| = \varphi(x)\,,\\ \mathcal{D}^+_{\hat\alpha}\mathcal{W}\| &= -\frac{\mathrm{i}}{8}\int \mathrm{d}u_1\,(u^+u^+_1)\,D^-_{1\hat\alpha}(\hat D^-_1)^2\,\mathcal{V}^{++}(z,u_1)\| = \mathrm{i}\,\Psi^i_{\hat\alpha}(x)\,u^+_i\,,\\ (\hat{\mathcal{D}}^+)^2\mathcal{W}\| &= \frac{\mathrm{i}}{8}\int \mathrm{d}u_1\,(u^+u^+_1)^2\,(\hat D^-_1)^2\,(\hat D^-_1)^2\,\mathcal{V}^{++}(z,u_1)\| = -4\mathrm{i}\,X^{ij}(x)\,u^+_iu^+_j\,.\end{aligned} \tag{2.39}$$

Finally, eq. (2.37) implies that the component field $F_{\hat\alpha\hat\beta} = F_{(\hat\alpha\hat\beta)}$ in (2.6) is (the bispinor form of) the gauge-covariant field strength $F_{\hat m\hat n}$ generated by the gauge field $A_{\hat m}$.

2.4 Fayet-Sohnius hypermultiplet

Following the four-dimensional $\mathcal{N} = 2$ supersymmetric construction due to Fayet and Sohnius [31, 32], an off-shell hypermultiplet with intrinsic central charge, which is coupled to the Yang-Mills supermultiplet, can be described by a superfield $\boldsymbol{q}^i(z)$ and its conjugate $\bar{\boldsymbol{q}}_i(z)$, $\bar{\boldsymbol{q}}_i = (\boldsymbol{q}^i)^\dagger$, subject to the constraint

$$\mathcal{D}^{(i}_{\hat\alpha}\,\boldsymbol{q}^{j)} = 0\,. \tag{2.40}$$

Introducing $\boldsymbol{q}^+(z,u) = \boldsymbol{q}^i(z)\, u_i^+$ and $\breve{\boldsymbol{q}}^+(z,u) = -\bar{\boldsymbol{q}}^i(z)\, u_i^+$, the constraint (2.40) can be rewritten in the form

$$\mathcal{D}^+_{\hat\alpha}\, \boldsymbol{q}^+ = \mathcal{D}^+_{\hat\alpha}\, \breve{\boldsymbol{q}}^+ = 0\,, \qquad \mathcal{D}^{++}\, \boldsymbol{q}^+ = \mathcal{D}^{++}\, \breve{\boldsymbol{q}}^+ = 0\ . \tag{2.41}$$

Thus $\boldsymbol{q}^+$ is a constrained analytic superfield. Using the algebra of gauge-covariant derivatives, the constraints can be shown to imply[6]

$$\begin{aligned}
&\widehat{\Box}\, \boldsymbol{q}^+ = 0\,, \qquad\qquad (2.42)\\
&\widehat{\Box} = \mathcal{D}^{\hat a}\mathcal{D}_{\hat a} + (\mathcal{D}^{+\hat\alpha}\mathcal{W})\, \mathcal{D}^-_{\hat\alpha} - \frac{1}{4}(\hat{\mathcal{D}}^{+\hat\alpha}\mathcal{D}^+_{\hat\alpha}\mathcal{W})\, \mathcal{D}^{--} + \frac{1}{8}[\mathcal{D}^{+\hat\alpha}, \mathcal{D}^-_{\hat\alpha}]\, \mathcal{W} + (\Delta + \mathrm{i}\,\mathcal{W})^2\ .
\end{aligned}$$

Therefore, the requirement of a constant central charge, $\Delta\, \boldsymbol{q}^+ = m\, \boldsymbol{q}^+$, with m a constant mass parameter, is equivalent to an equation of motion for the hypermultiplet.

Independent component fields of $\boldsymbol{q}^i(z)$ can be chosen as

$$C^i = \boldsymbol{q}^i\|\,, \qquad \lambda_{\hat\alpha} = \frac{\mathrm{i}}{\sqrt{8}}\, \mathcal{D}^i_{\hat\alpha}\, \boldsymbol{q}_i\|\,, \qquad F^i = \Delta \boldsymbol{q}^i\|\ . \tag{2.43}$$

All other components can be related to these and their derivatives. For example,

$$\begin{aligned}
(\hat{\mathcal{D}}^-)^2 \boldsymbol{q}^+ &= 8\mathrm{i}\, \Delta \boldsymbol{q}^- - 8\mathcal{W}\, \boldsymbol{q}^-\,,\\
\mathcal{D}^-_{\hat\alpha} \Delta \boldsymbol{q}^+ &= \mathcal{D}_{\hat\alpha}{}^{\hat\beta} \mathcal{D}^+_{\hat\beta}\, \boldsymbol{q}^- + \mathrm{i}\, \mathcal{W}\, \mathcal{D}^+_{\hat\alpha} \boldsymbol{q}^- + 2\mathrm{i}\, (\mathcal{D}^+_{\hat\alpha}\mathcal{W})\, \boldsymbol{q}^-\,,\\
(\hat{\mathcal{D}}^-)^2 \Delta \boldsymbol{q}^+ &= -8\mathrm{i}\, \mathcal{D}^{\hat a}\mathcal{D}_{\hat a} \boldsymbol{q}^- + 8\mathcal{W}\, \Delta \boldsymbol{q}^- + 8\mathrm{i}\, \mathcal{W}^2\, \boldsymbol{q}^- + 4\mathrm{i}\, (\mathcal{D}^{-\hat\alpha}\mathcal{W})\, \mathcal{D}^+_{\hat\alpha} \boldsymbol{q}^-\\
&\quad + 2\mathrm{i}\, (\mathcal{D}^{-\hat\alpha}\mathcal{D}^+_{\hat\alpha}\mathcal{W})\, \boldsymbol{q}^- - 2\mathrm{i}\, (\mathcal{D}^{-\hat\alpha}\mathcal{D}^-_{\hat\alpha}\mathcal{W})\, \boldsymbol{q}^+\ .
\end{aligned} \tag{2.44}$$

2.5 Off-shell hypermultiplets without central charge

One of the main virtues of the harmonic superspace approach [20] is that it makes possible an off-shell formulation for a charged hypermultiplet (transforming in an arbitrary representation of the gauge group) without central charge. Such a q^+-*hypermultiplet* is described by an unconstrained analytic superfield $q^+(z,u)$ and its conjugate $\breve{q}^+(z,u)$,

$$\mathcal{D}^+_{\hat\alpha}\, q^+ = 0\,, \qquad \Delta\, q^+ = 0\ . \tag{2.45}$$

In this approach, the requirement that q^+ be a holomorphic spinor field over the two-sphere, $\mathcal{D}^{++}\, q^+ = 0$, is equivalent to an equation of motion.[7] The harmonic dependence of the q^+-hypermultiplet is non-trivial. One can represent $q^+(z,u)$ by a convergent Fourier series of the form (B.9) with $p = 1$. The corresponding Fourier coefficients $q^{i_1\cdots i_{2n+1}}(z)$, where $n = 0, 1, \ldots$, obey some constraints that follow from the analyticity condition in (2.45).

[6] By analogy with the four-dimensional case [33], the operator $\widehat{\Box}$ can be called the covariant analytic d'Alembertian. Given a covariantly analytic superfield $\Phi^{(q)}$, the identity $\widehat{\Box}\Phi^{(q)} = -\frac{1}{64}(\hat{\mathcal{D}}^+)^2(\hat{\mathcal{D}}^+)^2(\mathcal{D}^{--})^2\Phi^{(q)}$ holds, and therefore $\widehat{\Box}$ preserves analyticity.

[7] The equation of motion for the massless Fayet-Sohnius hypermultiplet, which is characterised by the kinematic constraint $\mathcal{D}^{++}\, \boldsymbol{q}^+ = 0$, can be shown to be $\Delta\, \boldsymbol{q}^+ = 0$, if the dynamics is generated by the Lagrangian (3.16) with $m = 0$.

Given a hypermultiplet that transforms in a real representation of the gauge group, it can be described by a real anaytic superfied $\omega(z,u)$,

$$\mathcal{D}^+_{\hat{\alpha}}\,\omega = 0\,, \qquad \Delta\,\omega = 0\,, \tag{2.46}$$

called the *ω-hypermultiplet* [20]. The gauge parameter λ in (2.23) is of this superfield type. It is then clear that the ω-hypermultiplet can be used, for instance, to formulate a gauge-invariant Stückelberg description for massive vector multiplets.

3. Supersymmetric actions

In the case of vanishing central charge, $\Delta = 0$, it is easy to construct manifestly supersymmetric actions within the harmonic superspace approach [20]. Given a scalar harmonic superfield $L(z,u)$ and a scalar analytic superfield $L^{(+4)}(\zeta)$, supersymmetric actions are:

$$S_{\rm H} = \int {\rm d}^5x\,{\rm d}^8\theta\,{\rm d}u\,L = \int {\rm d}^5x\,{\rm d}u\,(\hat{D}^-)^4\,(\hat{D}^+)^4\,L\Big\|\,, \tag{3.1}$$

$$S_{\rm A} = \int {\rm d}\zeta^{(-4)}\,L^{(+4)} = \int {\rm d}^5x\,{\rm d}u\,(\hat{D}^-)^4\,L^{(+4)}\Big\|\,, \qquad \mathcal{D}^+_{\hat{\alpha}}L^{(+4)} = 0\,, \tag{3.2}$$

where

$$(\hat{D}^{\pm})^4 = -\frac{1}{32}(\hat{D}^{\pm})^2\,(\hat{D}^{\pm})^2\ . \tag{3.3}$$

As follows from (3.1), any integral over the full superspace can be reduced to an integral over the analytic subspace,

$$\int {\rm d}^5x\,{\rm d}^8\theta\,{\rm d}u\,L = \int {\rm d}\zeta^{(-4)}\,L^{(+4)}\,, \qquad L^{(+4)} = (\hat{D}^+)^4\,L\ . \tag{3.4}$$

The massless q^+-hypermultiplet action [20] is

$$S = -\int {\rm d}\zeta^{(-4)}\,\breve{q}^+\mathcal{D}^{++}q^+\ . \tag{3.5}$$

This action also describes a massive hypermutliplet if one assumes that (i) the gauge group is $G \times {\rm U}(1)$, and (ii) the U(1) gauge field $\mathcal{V}_0^{++}$ possesses a constant field strength $\mathcal{W}_0 = {\rm const}$, $|\mathcal{W}_0| = m$, see [34] for more details.[8]

Similarly to the chiral scalar in 4D, $\mathcal{N} = 1$ supersymmerty, couplings for the q^+-hypermultiplet are easy to construct. For example, one can consider a Lagrangian of the form

$$L^{(+4)} = -\breve{q}^+\mathcal{D}^{++}q^+ + \lambda\,(\breve{q}^+q^+)^2 + \breve{q}^+\Big\{\sigma_1\,(\hat{\mathcal{D}}^+)^2\mathcal{W} + {\rm i}\sigma_2\,\mathcal{G}^{++}\Big\}q^+\,, \tag{3.6}$$

with the quartic self-coupling first introduced in [20]. Consistent couplings for the Fayet-Sohnius hypermultiplet are much more restrictive, as a result of a non-vanishing intrinsic central charge.

[8]A different approach to formulate massive hypermultiplets was proposed in [35].

JHEP02(2006)038

3.1 Four-derivative vector multiplet action

As another example of supersymemtric action, we consider four-derivative couplings that may occur in low-energy effective actions for a U(1) vector multiplet.

$$S_{\text{four-deriv}} = \int \mathrm{d}\zeta^{(-4)}\, \mathcal{G}^{++}\Big\{\kappa_1\, \mathcal{G}^{++} + \mathrm{i}\,\kappa_2\, (\hat{D}^+)^2 \mathcal{W}\Big\} + \int \mathrm{d}^5x\, \mathrm{d}^8\theta\, H(\mathcal{W})\,, \tag{3.7}$$

with $\kappa_{1,2}$ coupling constants, the analytic superfield $\mathcal{G}^{++}$ given by (2.32), and $H(\mathcal{W})$ an arbitrary function. The third term on the right is a natural generalization of the four-derivative terms in 4D, $\mathcal{N}=2$ supersymmetry first introduced in [36].

3.2 Multiplets with intrinsic central charge

For 5D off-shell supermultiplets with $\Delta \neq 0$, the construction of supersymmetric actions is based on somewhat different ideas developed in [32, 37] for the case of 4D, $\mathcal{N}=2$ supersymmetric theories.

When $\Delta \neq 0$, there exists one more useful representation (in addition to the τ-frame and λ-frame) for the covariant derivatives

$$\mathcal{D}_{\mathbf{A}} \quad \mapsto \quad \boldsymbol{\nabla}_{\mathbf{A}} = \mathrm{e}^{\mathrm{i}(\Omega+\Sigma)}\, \mathcal{D}_{\mathbf{A}}\, \mathrm{e}^{-\mathrm{i}(\Omega+\Sigma)} \equiv \mathbf{D}_{\mathbf{A}} + \mathrm{i}\, \mathcal{V}_{\mathbf{A}}\,, \qquad \Sigma = -\theta^{-\hat{\alpha}}\theta^+_{\hat{\alpha}}\,\Delta\,. \tag{3.8}$$

For the operators $\mathbf{D}_{\mathbf{A}} = \mathrm{e}^{\mathrm{i}\Sigma}\, D_{\mathbf{A}}\, \mathrm{e}^{-\mathrm{i}\Sigma}$ one obtains

$$\begin{aligned} \boldsymbol{\nabla}^+_{\hat{\alpha}} &= \mathbf{D}^+_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{-\hat{\alpha}}}\,, \\ \mathbf{D}^{++} &= D^{++} + \mathrm{i}\,(\hat{\theta}^+)^2\,\Delta\,, \quad D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} + \mathrm{i}\,(\Gamma^{\hat{a}})_{\hat{\beta}\hat{\gamma}}\,\theta^{+\hat{\beta}}\theta^{+\hat{\gamma}}\frac{\partial}{\partial \boldsymbol{x}^{\hat{a}}} + \theta^{+\hat{\alpha}}\frac{\partial}{\partial\theta^{-\hat{\alpha}}}\,, \end{aligned} \tag{3.9}$$

where

$$(\hat{\theta}^+)^2 = \theta^{+\hat{\alpha}}\theta^+_{\hat{\alpha}}\,. \tag{3.10}$$

As is seen, in this frame the spinor gauge-covariant derivative $\boldsymbol{\nabla}^+_{\hat{\alpha}}$ coincides with partial derivatives with respect to $\theta^{-\hat{\alpha}}$, while the analyticity-preserving gauge-covariant derivative $\boldsymbol{\nabla}^{++} = \mathbf{D}^{++} + \mathrm{i}\,\mathcal{V}^{++}$ acquires a term proportional to the central charge.

In accordance with [37], the supersymmetric action involves a special gauge-invariant analytic superfield $\mathbf{L}^{++}$

$$\mathbf{D}^+_{\hat{\alpha}}\mathbf{L}^{++} = 0\,, \qquad \mathbf{D}^{++}\mathbf{L}^{++} = 0\,. \tag{3.11}$$

The action is

$$S = \mathrm{i}\int \mathrm{d}\zeta^{(-4)}\,(\hat{\theta}^+)^2\,\mathbf{L}^{++}\,. \tag{3.12}$$

Although S involves naked Grassmann variables, it turns out to be supersymmetric,due to the constraints imposed on $\mathbf{L}^{++}$. Its invariance under the supersymmetry transformations can be proved in complete analogy with the four-dimensional case [37]. The action (3.12) possesses another nice representation obtained in appendix C, eq. (C.13).

One can transform $\mathbf{L}^{++}$ to the τ-frame in which

$$L^{++}(z,u) = \mathrm{e}^{-\mathrm{i}\Sigma}\,\mathbf{L}^{++} = L^{ij}(z)\,u^+_i u^+_j\,. \tag{3.13}$$

This gauge-invariant superfield obeys the constrains

$$D^+_{\hat\alpha} L^{++} = 0\,, \qquad D^{++} L^{++} = 0\ . \tag{3.14}$$

Doing the Grassmann and harmonic integrals in (3.12) gives

$$S = \frac{\mathrm{i}}{12} \int \mathrm{d}^5 x\, \hat{\mathcal{D}}^{ij}\, L_{ij}\Big\|\,, \qquad \hat{\mathcal{D}}^{ij} = \mathcal{D}^{\hat\alpha(i}\mathcal{D}^{j)}_{\hat\alpha}\,, \tag{3.15}$$

where we have replaced, for convenience, ordinary spinor covariant derivatives by gauge-covariant ones (this obviously does not change the action, for L_{ij} is gauge invariant).

In four space-time dimensions, the super-action (3.15) was postulated by Sohnius [32] several years before the discovery of harmonic superspace. It is quite remarkable that only within the harmonic superspace approach, this super-action can be represented as a superspace integral having a transparent physical interpretation. To wit, the factor $\mathrm{i}(\hat\theta^+)^2$ in (3.12) can be identified with a vacuum expectation value $\langle \mathcal{V}^{++}_\Delta \rangle$ of the central-charge gauge superfield $\mathcal{V}^{++}_\Delta$ (compare with (2.39)). With such an interpretation, the super-action admits simple generalizations to the cases of (i) rigid supersymmetric theories with gauged central charge [38], and (ii) supergravity-matter systems [39].

The super-action (3.12), and its equivalent form (3.15), can be used for superymmetric theories without central charge; an example will be given below. It is only the constraints (3.14) which are relevant in the above construction.

3.3 Fayet-Sohnius hypermultiplet

An example of a theory with non-vanishing central charge is provided by the Fayet-Sohnius hypermultiplet. The Fayet-Sohnius hypermultiplet coupled to a Yang-Mills supermultiplet is described by the Lagrangian [32, 37]

$$L^{++}_{\rm FS} = \frac{1}{2} \breve{\boldsymbol{q}}^+ \overleftrightarrow{\Delta}\, \boldsymbol{q}^+ - \mathrm{i}\, m\, \breve{\boldsymbol{q}}^+ \boldsymbol{q}^+\,, \tag{3.16}$$

with m the hypermultiplet mass.

To compute the component action that follows from (3.16), one should use the definitions (2.6) and (2.43) for the component fields of $\mathcal{W}$ and $\boldsymbol{q}^i$, respectively.

$$\begin{aligned} S_{\rm FS} = \int \mathrm{d}^5 x \Big\{ &- \mathcal{D}_a \bar C_k \mathcal{D}^a C^k - \mathrm{i} \bar\lambda \not{\mathcal{D}} \lambda + \bar F_k F^k + m\, \bar\lambda\lambda + \bar\lambda \varphi \lambda - \frac{\mathrm{i}}{2} \bar C_k\, X^k{}_\ell\, C^\ell \\ &- \frac{1}{2} \bar C_k \varphi^2 C^k - \Big(\mathrm{i}\, m\, \bar F_k C^k - \frac{1}{\sqrt 8}\, \bar\lambda \Psi_k C^k + \mathrm{i}\, \bar F_k \varphi\, C^k + \mathrm{c.c.} \Big) \Big\}\ . \end{aligned} \tag{3.17}$$

3.4 Vector multiplet

The Yang-Mills supermultiplet is described by the Lagrangian[9]

$$L^{++}_{\rm YM} = \frac{1}{4} \mathrm{tr}\, \mathcal{G}^{++}\,, \qquad \Delta\, L^{++}_{\rm YM} = 0\,, \tag{3.18}$$

[9] In terms of the analytic prepotential $\mathcal{V}^{++}$, the super Yang-Mills action is non-polynomial [30].

with $\mathcal{G}^{++}$ given in (2.32). The corresponding equation of motion can be shown to be

$$(\hat{\mathcal{D}}^+)^2\mathcal{W} = 0 \quad \Leftrightarrow \quad \mathcal{D}^+_{\hat\alpha}\mathcal{D}^+_{\hat\beta}\mathcal{W} = 0 \ . \tag{3.19}$$

It follows from this that

$$\mathcal{D}^{\hat a}\mathcal{D}_{\hat a}\,\mathcal{W} = \frac{1}{2}\,\{\mathcal{D}^{\hat\alpha}_i\mathcal{W}\,,\mathcal{D}^i_{\hat\alpha}\mathcal{W}\} \ . \tag{3.20}$$

In the Abelian case, eq. (3.19) reduces to

$$D^+_{\hat\alpha}D^+_{\hat\beta}\mathcal{W} = 0 \quad \Rightarrow \quad \partial^{\hat a}\partial_{\hat a}\,\mathcal{W} = 0 \ . \tag{3.21}$$

From the point of view of 4D, $\mathcal{N} = 2$ supersymmetry, this can be recognized as the off-shell superfield constraints [40, 37, 41] describing the so-called linear vector-tensor multiplet discovered by Sohnius, Stelle and West [42] and re-vitalized fifteen years later in the context of superstring compactifications [43].

The Yang-Mills action with components defined by (2.34) is

$$\begin{aligned} S_{\rm YM} = \int {\rm d}^5x\,{\rm tr}\Big\{ -\frac{1}{4}\,F_{\hat a\hat b}F^{\hat a\hat b} - \frac{1}{2}\mathcal{D}_{\hat a}\varphi\mathcal{D}^{\hat a}\varphi + \frac{1}{4}X^{ij}X_{ij} \\ + \frac{\rm i}{2}\Psi^k\not{\mathcal{D}}\Psi_k - \frac{1}{2}\Psi^k\,[\varphi,\Psi_k]\Big\} \ . \end{aligned} \tag{3.22}$$

4. Chern-Simons couplings

Consider two vector multiplets: (i) a U(1) vector multiplet $\mathcal{V}^{++}_\Delta$; (ii) a Yang-Mills vector multilpet $\mathcal{V}^{++}_{\rm YM}$. They can be coupled to each other, in a gauge-invariant way, using the interaction

$$S_{\rm int} = \int {\rm d}\zeta^{(-4)}\,\mathcal{V}^{++}_\Delta\,{\rm tr}\,\mathcal{G}^{++}_{\rm YM}\,, \tag{4.1}$$

where $\mathcal{G}^{++}_{\rm YM}$ corresponds to the non-Abelian multiplet and is defined as in eq. (2.32). Invariance of $S_{\rm int}$ under the U(1) gauge transformations

$$\delta\mathcal{V}^{++} = -D^{++}\lambda\,, \qquad D^+_{\hat\alpha}\lambda = 0\,, \tag{4.2}$$

follows from the constraints (2.32) to which $\mathcal{G}^{++}_{\rm YM}$ is subject.

Let us assume that the physical scalar field in $\mathcal{V}^{++}_\Delta$ possesses a non-vanishing expectation value (such a situation occurs, for instance, when $\mathcal{V}^{++}_\Delta$ is the vector multiplet gauging the central charge symmetry). In accordance with [34], this condition is expressed as $\langle\mathcal{W}_\Delta(z)\rangle = \mu \neq 0$; then, there exists a gauge fixing such that

$$\langle\mathcal{V}^{++}_\Delta(\zeta,u)\rangle = {\rm i}\,\mu\,(\hat\theta^+)^2\,, \qquad \mu = {\rm const} \ . \tag{4.3}$$

Now, combining the interaction (4.1) with the gauge-invariant kinetic terms for $\mathcal{V}^{++}_\Delta$ and $\mathcal{V}^{++}_{\rm YM}$, the complete action becomes

$$S = \int {\rm d}\zeta^{(-4)}\,\mathcal{V}^{++}_\Delta\,\Big\{g_\Delta^{-2}\,\mathcal{G}^{++}_\Delta + g_{\rm YM}^{-2}\,{\rm tr}\,\mathcal{G}^{++}_{\rm YM}\Big\}\,, \tag{4.4}$$

with g_Δ and $g_{\rm YM}$ coupling constants. A different form for this action was given in [26]. The theory (4.4) is superconformal at the classical level. It would be interesting to compute, for instance, perturbative quantum corrections.

Let us consider the special case of a single Abelian gauge field, $\mathcal{V}^{++}_\Delta = \mathcal{V}^{++}_{\rm YM} \equiv \mathcal{V}^{++}$. The equations of motion for the corresponding Chern-Simons theory,

$$S_{\rm CS} = \frac{1}{12g^2}\int {\rm d}\zeta^{(-4)}\, \mathcal{V}^{++}\, \mathcal{G}^{++}\,, \tag{4.5}$$

can be shown to be

$$-{\rm i}\,\mathcal{G}^{++} = \mathcal{D}^{+\hat\alpha}\mathcal{W}\,\mathcal{D}^+_{\hat\alpha}\mathcal{W} + \frac{1}{2}\,\mathcal{W}\,(\hat{\mathcal{D}}^+)^2\mathcal{W} = 0\;. \tag{4.6}$$

Using the Bianchi identity (2.31), one can rewrite this in the form

$$D^+_{\hat\alpha}\,D^+_{\hat\beta}\,\mathcal{W} = -\frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\frac{D^{+\hat\gamma}\mathcal{W}\,D^+_{\hat\gamma}\mathcal{W}}{\mathcal{W}}\;. \tag{4.7}$$

From the point of view of 4D, $\mathcal{N}=2$ supersymmetry, this can be recognized as the off-shell superfield constraint describing the so-called nonlinear vector-tensor multiplet[10] [44, 38]. Resorting to the two-component spinor notation, eq. (4.7) leads to

$$D^+_\alpha \bar D^+_{\dot\alpha}\mathcal{W} = 0\,, \qquad D^{+\alpha}D^+_\alpha\mathcal{W} = -\frac{1}{\mathcal{W}}\Big(D^{+\alpha}\mathcal{W}\,D^+_\alpha\mathcal{W} - \bar D^+_{\dot\alpha}\mathcal{W}\,\bar D^{+\dot\alpha}\mathcal{W}\Big)\;. \tag{4.8}$$

In the case of the dynamical system (4.4), the equation of motion for the Abelian gauge field is

$$\frac{1}{\kappa}\,\mathcal{G}^{++}_\Delta = {\rm tr}\,\mathcal{G}^{++}_{\rm YM}\,, \tag{4.9}$$

with κ a coupling constant. With properly defined dimensional reduction 5D $\to$ 4D, this can be recognized as the superfield constraint describing the Chern-Simons coupling of a nonlinear vector-tensor to an external $\mathcal{N}=2$ Yang-Mills supermultiplet [38].

The super Chern-Simons actions can be readily reduced to components in the Wess-Zumino gauge (2.34) for the Abelian gauge field $\mathcal{V}^{++}$. If $L^{++}(z,u) = L^{ij}(z)\,u^+_i u^+_j$ is a real analytic superfield of the type (3.14), then

$$\begin{aligned} S &= \int {\rm d}\zeta^{(-4)}\,\mathcal{V}^{++}L^{++} \\ &= \int {\rm d}^5x\,\Big\{X^{ij}L_{ij} + \frac{\rm i}{12}\varphi\,\hat{\mathcal{D}}^{ij}L_{ij} + \frac{\rm i}{12}\,A_{\hat a}\,(\mathcal{D}^i\Gamma^{\hat a}\mathcal{D}^j)L_{ij} - \frac{2}{3}\,\Psi^{i\hat\alpha}\,\mathcal{D}^j_{\hat\alpha}L_{ij}\Big\}\Big\|\;. \end{aligned} \tag{4.10}$$

The Abelian supersymmetric Chern-Simons theory (4.5) leads to the following component action:

$$\begin{aligned} S_{\rm CS} = \frac{1}{2g^2}\int {\rm d}^5x\,\Big\{&\frac{1}{3}\,\epsilon^{\hat a\hat b\hat c\hat d\hat e}\,A_{\hat a}F_{\hat b\hat c}F_{\hat d\hat e} - \frac{1}{2}\,\varphi F_{\hat a\hat b}F^{\hat a\hat b} - \varphi\partial_{\hat a}\varphi\partial^{\hat a}\varphi + \frac{1}{2}\,\varphi X_{ij}X^{ij} \\ &-\frac{\rm i}{2}\,F_{\hat a\hat b}(\Psi^k\Sigma^{\hat a\hat b}\Psi_k) + {\rm i}\,\varphi(\Psi^k\,\partial\!\!\!/\,\Psi_k) - \frac{\rm i}{2}\,X_{ij}(\Psi^i\Psi^j)\Big\}\;. \end{aligned} \tag{4.11}$$

[10]The nonlinear vector-tensor multiplet was discovered in [45].

5. 5D supermultiplets in reduced superspace

Some of the results described in the previous sections can easily be reduced to a "hybrid" formulation which keeps manifest only 4D, $\mathcal{N}=1$ super Poincaré symmetry. As the 5D superfields depend on two sets of 4D anticommuting Majorana spinors, $(\theta^\alpha_{\underline{1}}, \bar\theta^{\underline{1}}_{\dot\alpha})$ and $(\theta^\alpha_{\underline{2}}, \bar\theta^{\underline{2}}_{\dot\alpha})$, such a hybrid formulation is equivalent to integrating out, say, the second set and keeping intact the first set of variables

$$\theta^\alpha = \theta^\alpha_{\underline{1}}\,, \qquad \bar\theta_{\dot\alpha} = \bar\theta^{\underline{1}}_{\dot\alpha}\,. \tag{5.1}$$

In this approach, one deals with reduced (or $\mathcal{N}=1$) superfields $U|$, $D^{\underline{2}}_\alpha U|$, $\bar D^{\dot\alpha}_{\underline{2}} U|, \dots$ (of which not all are usually independent) and 4D, $\mathcal{N}=1$ spinor covariant derivatives D_α and $\bar D^{\dot\alpha}$ defined in the obvious way:

$$U| = U(x, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})\Big|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}\,, \qquad D_\alpha = D^{\underline{1}}_\alpha\Big|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}\,, \qquad \bar D^{\dot\alpha} = \bar D^{\dot\alpha}_{\underline{1}}\Big|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}\,. \tag{5.2}$$

Our consideration below naturally reproduces many of the 5D supersymmetric models originally derived in the hybrid formulation [46].

5.1 Vector multiplet

Let us introduce reduced gauge covariant derivatives

$$\Big\{\boldsymbol{\mathcal{D}}_\alpha, \bar{\boldsymbol{\mathcal{D}}}^{\dot\alpha}, \boldsymbol{\mathcal{D}}_a, \boldsymbol{\mathcal{D}}_5\Big\} = \Big\{\mathcal{D}^{\underline{1}}_\alpha, \mathcal{D}^{\dot\alpha}_{\underline{1}}, \mathcal{D}_a, \mathcal{D}_5\Big\}\Big|\,. \tag{5.3}$$

As follows from (2.3), their algebra is

$$\begin{aligned}
\{\boldsymbol{\mathcal{D}}_\alpha\,, \boldsymbol{\mathcal{D}}_\beta\} = \{\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}, \bar{\boldsymbol{\mathcal{D}}}_{\dot\beta}\} = 0\,, \qquad & \{\boldsymbol{\mathcal{D}}_\alpha\,, \bar{\boldsymbol{\mathcal{D}}}_{\dot\beta}\} = -2{\rm i}\,\boldsymbol{\mathcal{D}}_{\alpha\dot\beta}\,,\\
[\boldsymbol{\mathcal{D}}_\alpha\,, \boldsymbol{\mathcal{D}}_{\beta\dot\beta}] = -2{\rm i}\varepsilon_{\alpha\beta}\,\bar{\mathcal{W}}_{\dot\beta}\,, \qquad & [\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\,, \boldsymbol{\mathcal{D}}_{\beta\dot\beta}] = -2{\rm i}\varepsilon_{\dot\alpha\dot\beta}\,\mathcal{W}_\beta\,,\\
[\boldsymbol{\mathcal{D}}_\alpha\,, \boldsymbol{\mathcal{D}}_5 + \mathcal{F}] = 0\,, \qquad & [\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\,, \boldsymbol{\mathcal{D}}_5 - \mathcal{F}] = 0\,,
\end{aligned} \tag{5.4}$$

where

$$\mathcal{F} = \mathcal{W}|\,, \qquad \mathcal{W}_\alpha = \mathcal{D}^{\underline{2}}_\alpha \mathcal{W}|\,. \tag{5.5}$$

It can be seen that the field strengths $\mathcal{F}$, $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}^{\dot\alpha}$ are the only independent $\mathcal{N}=1$ descendants of $\mathcal{W}$.

The strengths $\mathcal{F}$ and $\mathcal{W}_\alpha$ obey some constraints which follow from the Bianchi identities (2.4) and (2.5). Consider first the constraint (2.4) with two derivatives of $\mathcal{W}$. Taking the part with $(i, j, \hat\alpha, \hat\beta) = (\underline{1}, \underline{1}, \alpha, \dot\alpha)$ gives the "$\mathcal{N}=1$ chirality" of $\mathcal{W}_\alpha$

$$\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\mathcal{W}_\alpha = 0\,. \tag{5.6}$$

Taking instead the part with $(i, j, \hat\alpha, \hat\beta) = (\underline{1}, \underline{2}, \alpha, \beta)$ gives the familiar Bianchi identity

$$\boldsymbol{\mathcal{D}}^\alpha \mathcal{W}_\alpha - \bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} = 0\,. \tag{5.7}$$

Next, the $(i, j, \hat\alpha, \hat\beta) = (\underline{1}, \underline{1}, \alpha, \beta)$ and $(i, j, \hat\alpha, \hat\beta) = (\underline{1}, \underline{2}, \alpha, \dot\alpha)$ parts, respectively, give

$$\bar{\mathcal{D}}_{\underline{2}\dot\gamma}\bar{\mathcal{D}}^{\dot\gamma}_{\underline{2}}\mathcal{W}| = -\boldsymbol{\mathcal{D}}^2\mathcal{F}\,, \qquad \mathcal{D}^{\underline{2}}_{\alpha}\bar{\mathcal{D}}_{\underline{2}\dot\beta}\mathcal{W}| = \boldsymbol{\mathcal{D}}_\alpha\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\mathcal{F}\ . \tag{5.8}$$

The latter identities support the statement that $\mathcal{F}$, $\mathcal{W}_\alpha$ and $\bar{\mathcal{W}}^{\dot\alpha}$ are the only independent $\mathcal{N} = 1$ descendants of $\mathcal{W}$. Finally, decomposing the second constraint (2.5) with $(i, j, k) = (\underline{2}, \underline{2}, \underline{1})$ and $(\hat\alpha, \hat\beta, \hat\gamma) = (\dot\alpha, \dot\beta, \gamma)$ gives

$$-\frac{1}{4}\bar{\boldsymbol{\mathcal{D}}}^2\boldsymbol{\mathcal{D}}_\alpha\mathcal{F} + \boldsymbol{\mathcal{D}}_5\mathcal{W}_\alpha - [\mathcal{F}, \mathcal{W}_\alpha] = 0\ . \tag{5.9}$$

In accordance with (3.18), the super Yang-Mills action is

$$S_{\rm YM} = \frac{\rm i}{12}\int {\rm d}^5x\,\hat{\mathcal{D}}_{ij}L^{ij}_{\rm YM}\Big\|\,, \qquad L^{ij}_{\rm YM} = \frac{\rm i}{4}\,{\rm tr}\Big(\mathcal{D}^{\hat\alpha i}\mathcal{W}\,\mathcal{D}^j_{\hat\alpha}\mathcal{W} + \frac{1}{4}\{\mathcal{W}\,,\hat{\mathcal{D}}^{ij}\mathcal{W}\}\Big)\ . \tag{5.10}$$

Its reduced form can be shown to be

$$S_{\rm YM} = {\rm tr}\int {\rm d}^5x\,\left\{\frac{1}{4}\int {\rm d}^2\theta\,\mathcal{W}^\alpha\mathcal{W}_\alpha + \frac{1}{4}\int {\rm d}^2\bar\theta\,\bar{\mathcal{W}}_{\dot\alpha}\bar{\mathcal{W}}^{\dot\alpha} + \int {\rm d}^4\theta\,\mathcal{F}^2\right\}\ . \tag{5.11}$$

Here the Grassmann measures ${\rm d}^2\theta$ and ${\rm d}^4\theta$ are part of the chiral and the full superspace measures, respectively, in 4D, $\mathcal{N} = 1$ supersymmetric field theory.

It is instructive to solve the constraints encoded in (5.4). A general solution to the equations $\{\boldsymbol{\mathcal{D}}_\alpha\,, \boldsymbol{\mathcal{D}}_\beta\} = [\boldsymbol{\mathcal{D}}_\alpha\,, \boldsymbol{\mathcal{D}}_5 + \mathcal{F}] = 0$ is

$$\boldsymbol{\mathcal{D}}_\alpha = {\rm e}^{-\Xi}\,D_\alpha\,{\rm e}^{\Xi}\,, \qquad \boldsymbol{\mathcal{D}}_5 + \mathcal{F} = {\rm e}^{-\Xi}\left(\partial_5 + \Phi^\dagger\right){\rm e}^{\Xi}\,, \qquad D_\alpha\Phi^\dagger = 0\,, \tag{5.12}$$

for some Lie-algebra-valued prepotentials Ξ and $\Phi^\dagger$, of which Ξ is complex unconstrained and $\Phi^\dagger$ antichiral. Similarly, the constraints $\{\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}, \bar{\boldsymbol{\mathcal{D}}}_{\dot\beta}\} = [\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha}\,, \boldsymbol{\mathcal{D}}_5 - \mathcal{F}] = 0$ are solved by

$$\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha} = {\rm e}^{\Xi^\dagger}\,\bar D_{\dot\alpha}\,{\rm e}^{-\Xi^\dagger}\,, \qquad \boldsymbol{\mathcal{D}}_5 - \mathcal{F} = {\rm e}^{\Xi^\dagger}\left(\partial_5 - \Phi\right){\rm e}^{-\Xi^\dagger}\,, \qquad \bar D_{\dot\alpha}\Phi = 0\ . \tag{5.13}$$

The prepotentials introduced possess the following gauge transformations

$${\rm e}^{\Xi^\dagger} \ \to\ {\rm e}^{{\rm i}\tau(z)}\,{\rm e}^{\Xi^\dagger}\,{\rm e}^{-{\rm i}\lambda(z)}\,, \quad \Phi \ \to\ {\rm e}^{{\rm i}\lambda(z)}\left(\Phi - \partial_5\right){\rm e}^{-{\rm i}\lambda(z)}\cdot 1\,, \quad \bar D_{\dot\alpha}\lambda = 0\ . \tag{5.14}$$

Here the λ-gauge group occurs as a result of solving the constraints in terms of the unconstrained prepotentials.

By analogy with the 4D, $\mathcal{N} = 1$ super Yang-Mills case, one can introduce a chiral representation defined by applying a *complex* gauge transformation with $\tau = -\Xi^\dagger$. This gives

$$\begin{aligned} \boldsymbol{\mathcal{D}}_\alpha &= {\rm e}^{-V}\,D_\alpha\,{\rm e}^{V}\,, & \bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha} &= \bar D_{\dot\alpha}\,, \\ \boldsymbol{\mathcal{D}}_5 + \mathcal{F} &= {\rm e}^{-V}\left(\partial_5 + \Phi^\dagger\right){\rm e}^{V} & \boldsymbol{\mathcal{D}}_5 - \mathcal{F} &= \partial_5 - \Phi\,, \end{aligned} \tag{5.15}$$

where

$${\rm e}^V = {\rm e}^{\Xi}\,{\rm e}^{\Xi^\dagger}\,, \qquad V^\dagger = V\ .$$

Here the real Lie-algebra valued superfield V is the standard $\mathcal{N} = 1$ super Yang-Mills prepotential. For $\mathcal{F}$ we obtain

$$2\,\mathcal{F} = \Phi + {\rm e}^{-V}\,\Phi^\dagger\,{\rm e}^V + {\rm e}^{-V}\left(\partial_5\,{\rm e}^V\right)\ . \tag{5.16}$$

We have thus reproduced the results obtained by Hebecker within the hybrid approach [46].

JHEP02(2006)038

5.2 Fayet-Sohnius hypermultiplet

The Fayet-Sohnius hypermultiplet $\boldsymbol{q}^i$ generates two independent $\mathcal{N}=1$ superfields transforming in the same representation of the gauge group,

$$\tilde{Q}^\dagger = \boldsymbol{q}^{\underline{1}}| , \qquad Q = \boldsymbol{q}^{\underline{2}}| , \tag{5.17}$$

and obeying the constraints

$$\boldsymbol{\mathcal{D}}_\alpha \tilde{Q}^\dagger = 0 , \qquad \bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha} Q = 0 . \tag{5.18}$$

These constraints follow from (2.40). Thus Q and $\tilde{Q}^\dagger$ are covariantly chiral and antichiral, respectively. The central charge transformation of these superfields is:

$$\mathrm{i}\,\Delta Q = \frac{1}{4}\bar{\boldsymbol{\mathcal{D}}}^2 \tilde{Q}^\dagger + (\mathcal{F} - \boldsymbol{\mathcal{D}}_5) Q , \qquad \mathrm{i}\,\Delta \tilde{Q}^\dagger = \frac{1}{4}\boldsymbol{\mathcal{D}}^2 Q + (\mathcal{F} + \boldsymbol{\mathcal{D}}_5)\tilde{Q}^\dagger . \tag{5.19}$$

In accordance with (3.16), the action for the Fayet-Sohnius hypermultiplet is

$$S_{\mathrm{FS}} = \frac{\mathrm{i}}{12}\int \mathrm{d}^5 x\, \hat{\mathcal{D}}_{ij} L^{ij}_{\mathrm{FS}}\Big| , \qquad L^{ij}_{\mathrm{FS}} = -\Big(\frac{1}{2}\bar{\boldsymbol{q}}^{(i} \overset{\leftrightarrow}{\Delta} \boldsymbol{q}^{j)} - \mathrm{i}\, m \bar{\boldsymbol{q}}^{(i}\boldsymbol{q}^{j)}\Big) . \tag{5.20}$$

It can be shown to reduce to the following $\mathcal{N}=1$ action

$$S_{\mathrm{FS}} = \int \mathrm{d}^5 x \left\{ \int \mathrm{d}^4\theta\, (Q^\dagger Q + \tilde{Q}\,\tilde{Q}^\dagger) + \Big(\int \mathrm{d}^2\theta\, \tilde{Q}(\mathcal{F} - \boldsymbol{\mathcal{D}}_5 + m) Q + \mathrm{c.c.}\Big) \right\} . \tag{5.21}$$

As follows from (5.4), the operator $\boldsymbol{\mathcal{D}}_5 - \mathcal{F}$ preserves chirality.

6. Projective superspace and dimensional reduction

Throughout this section, we consider only 5D supermultiplets without central charge, $\Delta = 0$. However, many results below can be extended to include the case $\Delta \neq 0$.

6.1 Doubly punctured harmonic superspace

Let $\boldsymbol{\psi}^{(p)}(z,u)$ be a harmonic superfield of non-negative U(1) charge p. Here we will be interested in solutions to the equation

$$D^{++}\boldsymbol{\psi}^{(p)} = 0 \quad \Rightarrow \quad D^{++} D^+_{\hat\alpha}\boldsymbol{\psi}^{(p)} = 0 , \qquad p \geq 0 . \tag{6.1}$$

If $\boldsymbol{\psi}^{(p)}(z,u)$ is globally defined and smooth over $\mathbb{R}^{5|8} \times S^2$, it possesses a convergent Fourier series of the form (B.9). If $\boldsymbol{\psi}^{(p)}(z,u)$ is further constrained to obey the equation (6.1), then its general form becomes

$$\boldsymbol{\psi}^{(p)}(z,u) = \boldsymbol{\psi}^{i_1\cdots i_p}(z)\, u^+_{i_1}\cdots u^+_{i_p} . \tag{6.2}$$

Therefore, such a globally defined harmonic superfield possesses finitely many component fields, and this can thought of as a consequence of the Riemann-Roch theorem [47] specified to the case of S^2. A more interesting situation occurs if one allows $\boldsymbol{\psi}^{(p)}(z,u)$ to have a few singularities on S^2.

For further consideration, it is useful to cover S^2 by two charts and introduce local complex coordinates in each chart, as defined in appendix B. In the north chart (parametrized by the complex variable w and its conjugate $\bar{w}$) we can represent $\boldsymbol{\psi}^{(p)}(z,u)$ as follows

$$\boldsymbol{\psi}^{(p)}(z,u) = (u^{+\underline{1}})^p\, \psi(z,w,\bar{w})\ . \tag{6.3}$$

If $\boldsymbol{\psi}^{(p)}(z,u)$ is globally defined over $\mathbb{R}^{5|8}\times S^2$, then $\psi(z,w,\bar{w}) \equiv \boldsymbol{\psi}^{(p)}_{\mathrm{N}}(z,w,\bar{w})$ is given as in eq. (B.10). It is a simple exercise to check that

$$D^{++}\boldsymbol{\psi}^{(p)}(z,u) = (u^{+\underline{1}})^{p+2}\,(1+w\bar{w})^2\,\partial_{\bar{w}}\,\psi(z,w,\bar{w})\,, \qquad p\geq 0\,, \tag{6.4}$$

and therefore

$$D^{++}\,\boldsymbol{\psi}^{(p)} = 0\,, \quad p\geq 0 \quad \Leftrightarrow \quad \partial_{\bar{w}}\,\psi = 0\ . \tag{6.5}$$

Assuming that $\boldsymbol{\psi}^{(p)}(z,u)$ may possess singularities only at the north and south poles of S^2, we then conclude that

$$\psi(z,w) = \sum_{n=-\infty}^{+\infty} \psi_n(z)\, w^n\ . \tag{6.6}$$

Now, consider an analytic superfield $\boldsymbol{\phi}^{(p)}$ obeying the constraint (6.1).

$$D^+_{\hat{\alpha}}\,\boldsymbol{\phi}^{(p)} = 0\,, \qquad D^{++}\,\boldsymbol{\phi}^{(p)} = 0\,, \qquad p\geq 0\ . \tag{6.7}$$

We assume that $\boldsymbol{\phi}^{(p)}(z,u)$ is non-singular outside the north and south poles of S^2. Then, representing $\boldsymbol{\phi}^{(p)}(z,u) = (u^{+\underline{1}})^p\,\phi(z,w,\bar{w})$ and defining

$$D^+_{\hat{\alpha}} = -u^{+\underline{1}}\,\nabla_{\hat{\alpha}}(w)\,, \qquad \nabla_{\hat{\alpha}}(w) = -D^{\underline{i}}_{\hat{\alpha}}\,w_{\underline{i}}\,, \qquad w_{\underline{i}} = (-w,1)\,, \tag{6.8}$$

eq. (6.7) is solved as

$$\nabla_{\hat{\alpha}}(w)\,\phi(z,w) = 0\,, \qquad \phi(z,w) = \sum_{n=-\infty}^{+\infty}\phi_n(z)\,w^n\ . \tag{6.9}$$

These relations define a *projective superfield*, following the four-dimensional terminology [24]. Since the supersymmetry transformations act simply as the identity transformation on S^2, the above consideration clearly defines supermultiplets. Such supermultiplets turn out to be most suited for dimensional reduction.

The projective analogue of the smile-conjugation (2.21) is

$$\breve{\phi}(z,w) = \sum_{n=-\infty}^{+\infty}(-1)^n\,\bar{\phi}_{-n}(z)\,w^n\,, \qquad \nabla_{\hat{\alpha}}(w)\,\breve{\phi}(z,w) = 0\ . \tag{6.10}$$

If $\breve{\phi}(z,w) = \phi(z,w)$, the projective superfield is called real. The projective conjugation (6.10) can be derived from the smile-conjugation (2.21), see [25] for details.

There are several types of projective superfields [24]. A real projective superfield of the form (7.11) is called a *tropical* multiplet. A real projective superfield of the form

$$\phi(z,w) = \sum_{-n}^{+n} \phi_n(z)\, w^n\,, \qquad \breve{\phi} = \phi\,, \qquad\qquad n \in \mathbb{Z} \tag{6.11}$$

is called a real $\mathrm{O}(2n)$ multiplet.[11] A projective superfield $\Upsilon(z,w)$ of the form (6.27) is called an *arctic* multiplet, and its conjugate, $\breve{\Upsilon}(z,w)$, an *antarctic* multiplet. The $\Upsilon(z,w)$ and $\breve{\Upsilon}(z,w)$ constitute a *polar* multiplet. More general projective superfields occur if one multiplies any of the considered superfields by w^n, with n an integer.

At this stage, it is useful to break the manifest 5D Lorentz invariance by switching from the four-component spinor notation to the two-component one. Representing

$$\nabla_{\hat\alpha}(w) = \begin{pmatrix} \nabla_\alpha(w) \\ \bar\nabla^{\dot\alpha}(w) \end{pmatrix}, \quad \nabla_\alpha(w) \equiv w D^{\underline{1}}_\alpha - D^{\underline{2}}_\alpha\,, \quad \bar\nabla^{\dot\alpha}(w) \equiv \bar D^{\dot\alpha}_{\underline{1}} + w \bar D^{\dot\alpha}_{\underline{2}}\,, \tag{6.12}$$

the constraints (6.10) can be rewritten in the component form

$$D^{\underline{2}}_\alpha \phi_n = D^{\underline{1}}_\alpha \phi_{n-1}\,, \qquad \bar D^{\dot\alpha}_{\underline{2}} \phi_n = -\bar D^{\dot\alpha}_{\underline{1}} \phi_{n+1}\,. \tag{6.13}$$

In accordance with (A.18) and (A.19), one can think of the operators $D_A = (\partial_a, D^i_\alpha, \bar D^{\dot\alpha}_i)$, where $a = 0,1,2,3$, as the covariant derivatives of 4D, $\mathcal{N}=2$ central charge superspace, with x^5 being the central charge variable. The relations (6.13) imply that the dependence of the component superfields ϕ_n on $\theta^\alpha_{\underline{2}}$ and $\bar\theta^{\underline{2}}_{\dot\alpha}$ is uniquely determined in terms of their dependence on $\theta^\alpha_{\underline{1}}$ and $\bar\theta^{\underline{1}}_{\dot\alpha}$. In other words, the projective superfields depend effectively on half the Grassmann variables which can be choosen to be the spinor coordinates of 4D, $\mathcal{N}=1$ superspace (5.1). In other words, it is sufficient to work with reduced superfields $\phi(w)|$ and 4D, $\mathcal{N}=1$ spinor covariant derivatives D_α and $\bar D^{\dot\alpha}$ defined in (5.2).

If the series in (6.9) is bounded from below (above), then eq. (6.13) implies that the two lowest (highest) components in $\phi(w)|$ are constrained $\mathcal{N}=1$ superfields. For example, in the case of the arctic multiplet, eq. (6.27), the leading components $\Phi = \Upsilon_0|$ and $\Gamma = \Upsilon_1|$ obey the constraints (6.28).

Given a real projective superfield $L(z,w)$, one can construct a supersymmetric invariant

$$S = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}w}{w} \int \mathrm{d}^5x\, \mathrm{d}^4\theta\, L(w)\Big| \equiv \frac{1}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{d}w}{w}\, S(w)\,, \tag{6.14}$$

with C a contour around the origin (in what follows, such a contour is always assumed). For $S(w)$ there are several equivalent forms:

$$S(w) = \frac{1}{16} \int \mathrm{d}^5x\, D^2 \bar D^2\, L(z,w)\Big\| = \frac{1}{16} \int \mathrm{d}^5x\, (D^{\underline{1}})^2 (\bar D_{\underline{1}})^2\, L(z,w)\Big\| \tag{6.15}$$

assuming only that total space-time derivatives do not contribute. The invariance of $S(w)$ under arbitrary SUSY transformations is easy to demonstrate. Defining

$$D^4 = \frac{1}{16} (D^{\underline{1}})^2 (\bar D_{\underline{1}})^2\,, \tag{6.16}$$

[11] One can also introduce complex $\mathrm{O}(2n{+}1)$ multiplets [24].

one can argue as follows:

$$
\begin{aligned}
\delta S(w) &= \mathrm{i} \int \mathrm{d}^5 x \, \left(\varepsilon_i^\alpha Q_\alpha^i + \bar{\varepsilon}^i_{\dot{\alpha}} \bar{Q}_i^{\dot{\alpha}} \right) D^4 L(z,w) \Big\| = - \int \mathrm{d}^5 x \, \left(\varepsilon_i^\alpha D_\alpha^i + \bar{\varepsilon}^i_{\dot{\alpha}} \bar{D}_i^{\dot{\alpha}} \right) D^4 L(z,w) \Big\| \\
&= - \int \mathrm{d}^5 x \, \left(\varepsilon_{\underline{2}}^\alpha D_\alpha^{\underline{2}} + \bar{\varepsilon}^{\underline{2}}_{\dot{\alpha}} \bar{D}_{\underline{2}}^{\dot{\alpha}} \right) D^4 L(z,w) \Big\| = - \int \mathrm{d}^5 x \, D^4 \left(\varepsilon_{\underline{2}}^\alpha D_\alpha^{\underline{2}} + \bar{\varepsilon}^{\underline{2}}_{\dot{\alpha}} \bar{D}_{\underline{2}}^{\dot{\alpha}} \right) L(z,w) \Big\| \\
&= - \int \mathrm{d}^5 x \, D^4 \left(\varepsilon_2^\alpha D_\alpha^{\underline{1}} \, w - \bar{\varepsilon}^{\underline{2}}_{\dot{\alpha}} \bar{D}_{\underline{1}}^{\dot{\alpha}} \, w^{-1} \right) L(z,w) \Big\| = 0 \,,
\end{aligned}
\tag{6.17}
$$

with Q_α^i and $\bar{Q}_i^{\dot{\alpha}}$ the supersymmetry generators.

6.2 Tensor multiplet and nonlinear sigma-models

The tensor multiplet (also called O(2) multiplet) is described by a constrained real analytic superfield Ξ^{++}:

$$
D^+_{\hat{\alpha}} \, \Xi^{++} = 0 \,, \qquad D^{++} \, \Xi^{++} = 0 \ . \tag{6.18}
$$

The corresponding projective superfield $\Xi(z,w)$ is defined by $\Xi^{++}(z,u) = \mathrm{i} \, u^{+\underline{1}} u^{+\underline{2}} \, \Xi(z,w)$. Without distinguishing between $\Xi(z,w)$ and $\Xi(z,w)|$, we have

$$
\Xi(w) \;=\; \Phi + w \, G - w^2 \, \bar{\Phi} \,, \qquad \bar{G} = G \,, \tag{6.19}
$$

where the component superfields obey the constraints

$$
\bar{D}^{\dot{\alpha}} \, \Phi = 0 \,, \qquad -\frac{1}{4} \bar{D}^2 \, G = \partial_5 \, \Phi \ . \tag{6.20}
$$

Here we consider a 5D generalization of the 4D, $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model[12] studied in [48] and related to the so-called c-map [49]. Let F be a holomorphic function of n variables. Associated with this function is the following supersymmetric action

$$
S \;=\; - \int \mathrm{d}^5 x \, \mathrm{d}^4 \theta \, \Big[\frac{1}{2\pi \, \mathrm{i}} \oint \frac{\mathrm{d}w}{w} \, \frac{F(\Xi^I(w))}{w^2} \;+\; \text{c.c.} \Big] \ . \tag{6.21}
$$

Since

$$
\begin{aligned}
F(\Xi^I(w)) \;&=\; F\Big(\Phi^I + w \, G^I - w^2 \, \bar{\Phi}^I \Big) \\
&=\; F(\Phi) + w \, F_I(\Phi) G^I - w^2 \, \Big(F_I(\Phi) \bar{\Phi}^I - \frac{1}{2} \, F_{IJ}(\Phi) \, G^I G^J \Big) \;+\; O(w^3) \,,
\end{aligned}
$$

the contour integral is trivial to do. The action is equivalent to

$$
S[\Phi, \bar{\Phi}, G] = \int \mathrm{d}^5 x \, \mathrm{d}^4 \theta \, \Big\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} \, g_{I\bar{J}}(\Phi, \bar{\Phi}) \, G^I G^J \Big\} \,, \tag{6.22}
$$

where

$$
K(\Phi, \bar{\Phi}) = \bar{\Phi}^I \, F_I(\Phi) + \Phi^I \, \bar{F}_I(\bar{\Phi}) \,, \qquad g_{I\bar{J}}(\Phi, \bar{\Phi}) = F_{IJ}(\Phi) + \bar{F}_{IJ}(\bar{\Phi}) \ . \tag{6.23}
$$

The Kähler potential $K(\Phi, \bar{\Phi})$ generates the so-called rigid special Kähler geometry [50].

[12]The construction given in [48] has recently been reviewed and extended in [58].

Let us work out a dual formulation for the theory (6.22). Introduce a first-order action

$$
\begin{aligned}
&S[\Phi,\bar\Phi,G] + \int \mathrm{d}^5x \Big\{ \int \mathrm{d}^2\theta \, \Psi_I \Big(\partial_5 \Phi^I + \frac{1}{4}\bar D^2 G^I\Big) \ + \ \mathrm{c.c.} \Big\} \\
&= S[\Phi,\bar\Phi,G] - \int \mathrm{d}^5x \Big\{ \int \mathrm{d}^4\theta \, (\Psi_I + \bar\Psi_I)\, G^I + \Big(\int \mathrm{d}^2\theta \, \Psi_I \partial_5 \Phi^I \ + \ \mathrm{c.c.}\Big)\Big\} , \qquad (6.24)
\end{aligned}
$$

where the superfield G^I is now real unconstrained, while Ψ_I is chiral, $\bar D_{\dot\alpha}\Psi_I = 0$. In this action we can integrate out G^I using the corresponding equations of motion. This gives

$$
S[\Phi,\bar\Phi,\Psi,\bar\Psi] = \int \mathrm{d}^5x \Big\{ \int \mathrm{d}^4\theta \, H(\Phi,\bar\Phi,\Psi,\bar\Psi) + \Big(\int \mathrm{d}^2\theta \, \Psi_I \partial_5 \Phi^I \ + \ \mathrm{c.c.}\Big)\Big\} , \qquad (6.25)
$$

where

$$
H(\Phi,\bar\Phi,\Psi,\bar\Psi) = K(\Phi,\bar\Phi) + \frac{1}{2} g^{I\bar J}(\Phi,\bar\Phi)(\Psi_I + \bar\Psi_I)(\Psi_J + \bar\Psi_J) \ . \qquad (6.26)
$$

The potential $H(\Phi,\bar\Phi,\Psi,\bar\Psi)$ is the Kähler potential of a hyper Kähler manifold. By construction, this potential is generated by another Kähler potential, $K(\Phi,\bar\Phi)$, which is associated with the holomorphic function $F(\Phi)$ defining the rigid special Kähler geometry [50]. The correspondence $K(\Phi,\bar\Phi) \ \to \ H(\Phi,\bar\Phi,\Psi,\bar\Psi)$ is called the rigid c-map [49].

6.3 Polar hypermultiplet and nonlinear sigma-models

According to [24], the polar hypermultiplet is generated by projective superfields

$$
\Upsilon(z,w) = \sum_{n=0}^{\infty} \Upsilon_n(z)\, w^n , \qquad \breve\Upsilon(z,w) = \sum_{n=0}^{\infty} (-1)^n \bar\Upsilon_n(z)\, \frac{1}{w^n} \ . \qquad (6.27)
$$

The projective superfields Υ and $\breve\Upsilon$ are called arctic and antarctic [24], respectively. The constraints (6.13) imply that the leading components $\Phi = \Upsilon_0|$ and $\Gamma = \Upsilon_1|$ are constrained

$$
\bar D^{\dot\alpha} \Phi = 0 , \qquad -\frac{1}{4}\bar D^2 \Gamma = \partial_5 \Phi \ . \qquad (6.28)
$$

The other components of $\Upsilon(w)$ are complex unconstrained superfields, and they appear to be non-dynamical (auxiliary) in models with at most two space-time derivatives at the component level.

Here we consider a 5D generalization of the 4D, $\mathcal{N} = 2$ supersymmetric nonlinear sigma-model studied in [51]. It is described by the action

$$
S[\Upsilon,\breve\Upsilon] = \int \mathrm{d}^5x \, \mathrm{d}^4\theta \, \Big[\frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}w}{w} \, K\big(\Upsilon(w),\breve\Upsilon(w)\big) \Big] \ . \qquad (6.29)
$$

This 5D supersymmetric sigma-model respects all the geometric features of its 4D, $\mathcal{N} = 1$ predecessor [52]

$$
S[\Phi,\bar\Phi] = \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, K(\Phi, \bar\Phi) , \qquad (6.30)
$$

where $K(A,\bar A)$ is the Kähler potential of some manifold $\mathcal{M}$. The Kähler invariance of (6.30)

$$
K(\Phi,\bar\Phi) \quad \longrightarrow \quad K(\Phi,\bar\Phi) \ + \ \Big(\Lambda(\Phi) \ + \ \bar\Lambda(\bar\Phi)\Big) \qquad (6.31)
$$

turns into

$$K(\Upsilon, \breve{\Upsilon}) \quad \longrightarrow \quad K(\Upsilon, \breve{\Upsilon}) \; + \; \Big(\Lambda(\Upsilon) + \bar{\Lambda}(\breve{\Upsilon})\Big) \tag{6.32}$$

for the model (6.29). A holomorphic reparametrization $A^I \mapsto f^I\big(A\big)$ of the Kähler manifold has the following counterparts

$$\Phi^I \quad \mapsto \quad f^I\big(\Phi\big)\,, \qquad\qquad \Upsilon^I(w) \quad \mapsto \quad f^I\big(\Upsilon(w)\big) \tag{6.33}$$

in the 4D and 5D cases, respectively. Therefore, the physical superfields of the 5D theory

$$\Upsilon^I(w)\Big|_{w=0} \; = \; \Phi^I\,, \qquad\qquad \frac{\mathrm{d}\Upsilon^I(w)}{\mathrm{d}w}\Big|_{w=0} \; = \; \Gamma^I\,, \tag{6.34}$$

should be regarded, respectively, as a coordinate of the Kähler manifold and a tangent vector at point Φ of the same manifold. That is why the variables (Φ^I, Γ^J) parametrize the tangent bundle $T\mathcal{M}$ of the Kähler manifold $\mathcal{M}$.

The auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, can be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \mathrm{d}w\, w^{n-1}\, \frac{\partial K(\Upsilon, \breve{\Upsilon})}{\partial \Upsilon^I} = 0\,, \qquad n \geq 2\;. \tag{6.35}$$

Their elimination can be carried out using the ansatz [53]

$$\Upsilon^I_n = \sum_{p=o}^{\infty} G^I{}_{J_1 \dots J_{n+p}\, \bar{L}_1 \dots \bar{L}_p}(\Phi, \bar{\Phi})\, \Gamma^{J_1} \dots \Gamma^{J_{n+p}}\, \bar{\Gamma}^{\bar{L}_1} \dots \bar{\Gamma}^{\bar{L}_p}\,, \qquad n \geq 2\;. \tag{6.36}$$

Upon elimination of the auxiliary superfields,[13] the action (6.29) takes the form

$$\begin{aligned} S_{\rm tb}[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}] = \int \mathrm{d}^5x\, \mathrm{d}^4\theta \,\Big\{\, & K\big(\Phi, \bar{\Phi}\big) - g_{I\bar{J}}\big(\Phi, \bar{\Phi}\big)\Gamma^I\bar{\Gamma}^{\bar{J}} \\ & + \sum_{p=2}^{\infty} \mathcal{R}_{I_1 \cdots I_p \bar{J}_1 \cdots \bar{J}_p}\big(\Phi, \bar{\Phi}\big)\Gamma^{I_1} \dots \Gamma^{I_p}\bar{\Gamma}^{\bar{J}_1} \dots \bar{\Gamma}^{\bar{J}_p} \,\Big\}\,, \end{aligned} \tag{6.37}$$

where the tensors $\mathcal{R}_{I_1 \cdots I_p \bar{J}_1 \cdots \bar{J}_p}$ are functions of the Riemann curvature $R_{I\bar{J}K\bar{L}}\big(\Phi, \bar{\Phi}\big)$ and its covariant derivatives. Each term in the action contains equal powers of Γ and $\bar{\Gamma}$, since the original model (6.29) is invariant under rigid U(1) transformations

$$\Upsilon(w) \quad \mapsto \quad \Upsilon(\mathrm{e}^{\mathrm{i}\alpha} w) \quad \Longleftrightarrow \quad \Upsilon_n(z) \quad \mapsto \quad \mathrm{e}^{\mathrm{i}n\alpha}\Upsilon_n(z)\;. \tag{6.38}$$

For the theory with action $S_{\rm tb}[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}]$, we can develop a dual formulation involving only chiral superfields and their conjugates as the dynamical variables. Consider the first-order action

$$\begin{aligned} & S_{\rm tb}[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}] - \int \mathrm{d}^5x\, \Big\{ \int \mathrm{d}^2\theta\, \Psi_I \Big(\partial_5\, \Phi^I + \frac{1}{4}\bar{D}^2\, \Gamma^I\Big) \; + \; \text{c.c.}\Big\} \\ & = S_{\rm tb}[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}] + \int \mathrm{d}^5x\, \Big\{ \int \mathrm{d}^4\theta\, \Psi_I\, \Gamma^I - \int \mathrm{d}^2\theta\, \Psi_I \partial_5\, \Phi^I \; + \; \text{c.c.}\Big\}\,, \end{aligned} \tag{6.39}$$

[13] As explained in [51], the auxiliary superfields can be eliminated only perturbatively for general Kähler manifolds. This agrees with a theorem proved in [54] that, for a Kähler manifold $\mathcal{M}$, a canonical hyper-Kähler structure exists, in general, on an open neighborhood of the zero section of the cotangent bundle $T^*\mathcal{M}$. It was further demonstrated in [51] that the auxiliary superfields can be eliminated in the case of compact Kähler symmetric spaces.

where the tangent vector Γ^I is now complex unconstrained, while the one-form Ψ_I is chiral, $\bar{D}_{\dot{\alpha}}\Psi_I = 0$. Upon elimination of Γ and $\bar{\Gamma}$, with the aid of their equations of motion, the action turns into $S_{\rm cb}[\Phi, \bar{\Phi}, \Psi, \bar{\Psi}]$. Its target space is the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M}$.

It is instructive to consider a free hypermultiplet described by the Kähler potential $K_{\rm free}(A, \bar{A}) = \bar{A}\, A$. Then

$$S_{\rm free}[\Upsilon, \breve{\Upsilon}] = \int {\rm d}^5x\, {\rm d}^4\theta \sum_{n=0}^{\infty} (-1)^n \bar{\Upsilon}_n(z) \Upsilon_n = \int {\rm d}^5x\, {\rm d}^4\theta \left(\bar{\Phi}\,\Phi - \bar{\Gamma}\,\Gamma\right) \;+\; \dots \tag{6.40}$$

Here the dots stand for the auxiliary superfields' contribution. Now, eliminating the auxiliary superfields and dualizing Γ into a chiral scalar, one obtains the action for the free Fayet-Sohnius hypermultiplet, equation (5.21).

7. Vector multiplet in projective superspace

In the Abelian case, the gauge transformation (2.33) simplifies

$$\delta \mathcal{V}^{++} = -D^{++}\lambda\,, \qquad D^+_{\hat{\alpha}}\lambda = 0\,, \qquad \breve{\lambda} = \lambda\,. \tag{7.1}$$

The field strength (2.30) also simplifies

$$\mathcal{W} = \frac{\rm i}{8} \int {\rm d}u\, (\hat{D}^-)^2\, \mathcal{V}^{++}\,. \tag{7.2}$$

It is easy to see that $\mathcal{W}$ is gauge invariant.

The gauge freedom (7.1) can be used to choose the supersymmetric Lorentz gauge [20]

$$D^{++}\mathcal{V}^{++} = 0\,. \tag{7.3}$$

In other words, in this gauge $\mathcal{V}^{++}$ becomes a real O(2) multiplet,

$$\mathcal{V}^{++} = {\rm i}\, u^{+\underline{1}} u^{+\underline{2}}\, V(z,w)\,, \qquad V(z,w) = \frac{1}{w}\,\varphi(z) + V(z) - w\,\bar{\varphi}(z)\,. \tag{7.4}$$

Since $\mathcal{W}$ is gauge invariant, for its evaluation one can use any potential $\mathcal{V}^{++}$ from the same gauge orbit, in particular the one obeying the gauge condition (7.3). This Lorentz gauge is particularly useful for our consideration. Using the relation (C.6) and noting that $|u^{+\underline{1}}|^2 = (1 + w\bar{w})^{-1}$, we can rewrite $\mathcal{W}$ in the form

$$\mathcal{W} = \frac{1}{2} \int {\rm d}u\, \mathcal{P}(w)\, V(z,w)\,. \tag{7.5}$$

This can be further transformed to

$$\mathcal{W} = \frac{1}{4\pi{\rm i}} \oint \frac{{\rm d}w}{w}\, \mathcal{P}(w)\, V(z,w)\,. \tag{7.6}$$

Indeed, the consideration in appendix C justifies the following identity

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \int {\rm d}u\, \phi_{R,\epsilon}(u) = \frac{1}{2\pi{\rm i}} \oint \frac{{\rm d}w}{w}\, \phi(w)\,, \tag{7.7}$$

with the regularization $\phi_{R,\epsilon}(u) = \phi_{R,\epsilon}(w, \bar{w})$ of a function $\phi(w)$ holomorphic on $\mathbb{C}^*$ defined according to (C.2). Since the integrand on the right of (7.5) is, by construction, a smooth scalar field on S^2, we obvoiusly have

$$\int \mathrm{d}u\, \mathcal{P}(w)\, V(z,w) = \lim_{R\to\infty} \lim_{\epsilon\to 0} \int \mathrm{d}u\, \mathcal{P}(w)\, V_{R,\epsilon}(z,u)\ . \tag{7.8}$$

The representation (7.6) allows one to obtain a new formulation for the vector multiplet. Let $\Lambda(z,w)$ be an arctic multiplet

$$\Lambda(z,w) = \sum_{n=0}^{\infty} \Lambda_n(z)\, w^n\,, \qquad \nabla_{\hat{\alpha}}(w)\Lambda(z,w) = 0\,, \tag{7.9}$$

and $\breve{\Lambda}(z,w)$ its smile-conjugate. It then immediately follows that

$$\oint \frac{\mathrm{d}w}{w}\, \mathcal{P}(w)\, \Lambda(z,w) = \oint \frac{\mathrm{d}w}{w}\, \mathcal{P}(w)\breve{\Lambda}(z,w) = 0\ . \tag{7.10}$$

Now, introduce a real tropical multiplet $V(z,w)$,

$$V(z,w) = \sum_{n=-\infty}^{+\infty} V_n(z)\, w^n\,, \qquad \nabla_{\hat{\alpha}}(w)V(z,w) = 0\,, \qquad \bar{V}_n = (-1)^n\, V_{-n}\,, \tag{7.11}$$

possessing the gauge freedom

$$\delta V(z,w) = \mathrm{i}\Big(\breve{\Lambda}(z,w) - \Lambda(z,w)\Big)\ . \tag{7.12}$$

With such gauge transformations, eq. (7.6) defines a gauge invariant field strength. Next, in accordance with the superfield structure of the tropical and arctic multiplets, the gauge freedom can be used to turn $V(z,w)$ into a real O(2) multiplet, i.e. to bring $V(z,w)$ to the form (7.4). We thus arrive at the projective superspace formulation[14] for the vector multiplet [24].

Now, we are in a position to evaluate the $\mathcal{N} = 1$ field strengths (5.5) in terms of the prepotentials V_n. It follows from (C.6) that

$$\mathcal{F} = \mathcal{W}| = \frac{1}{4\pi\mathrm{i}} \oint \frac{\mathrm{d}w}{w}\mathcal{P}(w)\, V(w)\Big| = \frac{1}{2}\left(\Phi + \bar{\Phi} + \partial_5 V\right)\,, \tag{7.13}$$

where we have defined

$$\Phi = \frac{1}{4}\, \bar{D}^2 V_1|\,, \qquad V = V_0|\ . \tag{7.14}$$

The spinor field strength $\mathcal{W}_\alpha$ is given by

$$\mathcal{W}_\alpha(z) = D^{\underline{2}}_\alpha \mathcal{W}| = \frac{1}{4\pi\mathrm{i}} \oint \frac{\mathrm{d}w}{w} \left([D^{\underline{2}}_\alpha, \mathcal{P}(w)] + \mathcal{P}(w) D^{\underline{2}}_\alpha\right) V(w)\Big|\ . \tag{7.15}$$

[14] An alternative procedure to deduce the projective superspace formulation for the 4D, $\mathcal{N} = 2$ vector multiplet from the corresponding harmonic superspace formulation can be found in [25].

However, as $[D^{\underline{2}}_{\alpha}, \mathcal{P}(w)] = w\,\partial_5 D^{\underline{1}}_{\alpha}$ and given that for any projective superfield $\phi(w)$ we have $D^{\underline{2}}_{\alpha}\phi(w) = wD^{\underline{1}}_{\alpha}\phi(w)$, this expression reduces to

$$\mathcal{W}_{\alpha} = \frac{1}{4\pi \mathrm{i}} \oint \frac{\mathrm{d}w}{w} \left(\frac{1}{4}\bar{D}^2 D_{\alpha}\right) V(w)\Big| = \frac{1}{8}\bar{D}^2 D_{\alpha} V \ . \tag{7.16}$$

It can be seen that the gauge transformation (7.12) acts on the superfields in (7.14) as follows:

$$\delta V = \mathrm{i}\,(\bar{\Lambda} - \Lambda)\,, \qquad \delta\Phi = \mathrm{i}\,\partial_5\,\Lambda \qquad\qquad \Lambda = \Lambda_1| \ . \tag{7.17}$$

The approach presented in this section can be applied to reformulate the supersymmetric Chern-Simons theory (4.5) in projective superspace, and the possibility for this is based on the following observation. Let $\mathcal{L}^{++}$ be a linear multiplet, that is a real analytic superfield obeying the constraint $D^{++}\,\mathcal{L}^{++} = 0$. Then, the functional

$$\int \mathrm{d}\zeta^{(-4)}\,\mathcal{V}^{++}\,\mathcal{L}^{++}$$

is invariant under the gauge transformations (7.1). We can further represent $\mathcal{L}^{++} = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})L(z,w)$, where $\mathcal{L}(z,w)$ is a real O(2) multiplet. Then the functional

$$-\frac{1}{2\pi\,\mathrm{i}} \int \mathrm{d}^5x\,\mathrm{d}^4\theta\, \oint \frac{\mathrm{d}w}{w}\, V(w)\, L(w)\Big|$$

is invariant under the gauge transformations (7.12).

In the case of Chern-Simons theory (4.5), the role of $\mathcal{L}^{++}$ is played by the gauge-invariant superfield $(12g^2)^{-1}\,\mathcal{G}^{++}$, with $\mathcal{G}^{++}$ defined in (2.32). With the real O(2) multiplet $G(z,w)$ introduced by

$$\mathcal{G}^{++} = (\mathrm{i}u^{+\underline{1}}u^{+\underline{2}})\,G(z,w)\,, \qquad G(w) = -\frac{1}{w}\,\Psi + K + w\,\bar{\Psi}\,, \tag{7.18}$$

the Chern-Simons theory (4.5) is equivalently described by the action

$$12g^2\,S_{\mathrm{CS}} = -\frac{1}{2\pi\,\mathrm{i}} \int \mathrm{d}^5x\,\mathrm{d}^4\theta\, \oint \frac{\mathrm{d}w}{w}\, V(w)\, G(w)\Big| \equiv 12g^2 \int \mathrm{d}^5x\, \boldsymbol{\mathcal{L}}_{\mathrm{CS}} \ . \tag{7.19}$$

Direct evaluation of Ψ and K gives

$$\begin{aligned} \Psi &= -\mathcal{W}^{\alpha}\mathcal{W}_{\alpha} + \frac{1}{2}\bar{D}^2(\mathcal{F}^2)\,, \\ K &= -\mathcal{F}D^{\alpha}\mathcal{W}_{\alpha} - 2(D^{\alpha}\mathcal{F})\mathcal{W}_{\alpha} + \mathrm{c.c.} + 2\partial_5(\mathcal{F}^2) \ . \end{aligned} \tag{7.20}$$

These results lead to

$$\begin{aligned} 12g^2\,\boldsymbol{\mathcal{L}}_{\mathrm{CS}} = \int \mathrm{d}^2\theta\,\Phi\,\mathcal{W}^{\alpha}\mathcal{W}_{\alpha} + \int \mathrm{d}^4\theta\, V\,\left[\mathcal{F}D^{\alpha}\mathcal{W}_{\alpha} + 2(D^{\alpha}\mathcal{F})\mathcal{W}_{\alpha}\right] \ + \ \mathrm{c.c.} \\ + 4\int \mathrm{d}^4\theta\,\mathcal{F}^3 \ . \end{aligned} \tag{7.21}$$

Here we have chosen to present the answer in the form (potential) × (fieldstrength) × (fieldstength) analogously to the standard representation of the bosonic Chern-Simons action.[15]

The structure of the superspace action obtained is the following. The first and second line of (7.21) are separately invariant under the gauge transformation (7.17) up to surface terms as is easily seen. The relative factor of 4 is fixed by five-dimensional Lorentz invariance. This could be derived either from the component projection or, less painfully, by checking the five-dimensional mass-shell condition on the super-fieldstrengths using their equations of motion together with their Bianchi identities. Finally, under the shift $\Phi \mapsto \Phi + 1$, the action shifts by $S_{\rm CS} \mapsto S_{\rm CS} + S_{\rm YM} +$ surface term, where $S_{\rm YM}$ is the 5D Yang-Mills action (5.11) with the proper normalization.

For completeness, we also present here projective superspace extensions of the vector multiplet mass term and the Fayet-Iliopoulos term (their harmonic superspace form is given in [20]). The vector multiplet mass term is

$$-m^2 \int {\rm d}\zeta^{(-4)}\, (\mathcal{V}^{++})^2 \quad \longrightarrow \quad \frac{m^2}{2\pi\, {\rm i}} \int {\rm d}^5x\, {\rm d}^4\theta \oint \frac{{\rm d}w}{w}\, V^2(w)\Big| \ . \tag{7.22}$$

The gauge invariant Fayet-Iliopoulos term is

$$\int {\rm d}\zeta^{(-4)}\, c^{++}\, \mathcal{V}^{++} \quad \longrightarrow \quad -\frac{1}{2\pi\, {\rm i}} \int {\rm d}^5x\, {\rm d}^4\theta \oint \frac{{\rm d}w}{w}\, c(w)V(w)\Big| \ , \tag{7.23}$$

where $c^{++} = c^{ij} u^+_i u^+_j$, with a constant real iso-vector c^{ij}. Defining $c^{++} = {\rm i} u^{+\underline{1}} u^{+\underline{2}}\, c(w)$, with $c(w) = w^{-1}\bar{\xi}_{\mathbb{C}} + \xi_{\mathbb{R}} - w\xi_{\mathbb{C}}$, the FI action then reduces to

$$\xi_{\mathbb{R}} \int {\rm d}^5x\, {\rm d}^4\theta\, V + 2{\rm Re}\left(\xi_{\mathbb{C}} \int {\rm d}^5x\, {\rm d}^2\theta\, \Phi\right) \ . \tag{7.24}$$

So far the considerations in this section have been restricted to the Abelian case. It is necessary to mention that the projective superspace approach [24] can be generalized to provide an elegant description of 5D super Yang-Mills theories, which is very similar to the well-known description of 4D, $\mathcal{N} = 1$ supersymmetric theories. In particular, the Yang-Mills supermultiplet is described by a real Lie-algebra-valued tropical superfield $V(z, w)$ with the gauge transformation

$${\rm e}^{V(w)} \ \to \ {\rm e}^{{\rm i}\breve{\Lambda}(w)}\, {\rm e}^{V(w)}\, {\rm e}^{-{\rm i}\Lambda(w)} \ , \tag{7.25}$$

which is the non-linear generalization of the Abelian gauge transformation (7.12). The hypermultiplet sector is described by an arctic superfield $\Upsilon(z, w)$ and its conjugate, with the gauge transformation

$$\Upsilon(w) \ \to \ {\rm e}^{{\rm i}\Lambda(w)}\, \Upsilon(w) \ . \tag{7.26}$$

The hypermultiplet gauge-invariant action is

$$S[\Upsilon, \breve{\Upsilon}, V] = \frac{1}{2\pi {\rm i}} \oint \frac{{\rm d}w}{w} \int {\rm d}^5x\, {\rm d}^4\theta\, \breve{\Upsilon}(w)\, {\rm e}^{V(w)}\, \Upsilon(w) \ \ . \tag{7.27}$$

[15]The result presented here was given previously in the first reference of [46]. In comparing the results one should keep in mind that terms such as $\int {\rm d}^4\theta\, V\left[\frac{1}{2}(\Phi + \bar{\Phi})D^\alpha \mathcal{W}_\alpha + D^\alpha \Phi \mathcal{W}_\alpha\right] +$ c.c. can be rewritten to look like $\int {\rm d}^2\theta\, \Phi \mathcal{W}^\alpha \mathcal{W}_\alpha +$ c.c., thereby changing the appearance of the action.

8. Conclusion

In the present paper we have developed the manifestly supersymmetric approach to five-dimensional globally supersymmetric gauge theories. It is quite satisfying that 5D superspace techniques provide a universal setting to formulate all such theories in a compact, transparent and elegant form, similarly to the four-dimensional $\mathcal{N}=1$ and $\mathcal{N}=2$ theories. We believe that these techniques are not only elegant but, more importantly, are useful. In particular, these techniques may be useful for model building in the context of supersymmetric brane-world scenarios. The two examples of supersymmetric nonlinear sigma-models, which were constructed in section 6, clearly demonstrate the power of the 5D superspace approach.

Five-dimensional super Yang-Mills theories possess interesting properties at the quantum level [55]. Further insight into their quantum mechanical structure may be obtained by carrying out explicit supergraph calculations. Supersymmetric Chern-Simons theories (4.4) are truly interesting in this respect.

Note added. After this paper was posted to the hep-th archive, we were informed of a related interesting work on 6D, $\mathcal{N}=(1,0)$ supersymmetric field theories [57].

Acknowledgments

S.M.K. is grateful to Jim Gates and the Center for String and Particle Theory at the University of Maryland, where this project was conceived, for hospitality. The work of S.M.K. is supported in part by the Australian Research Council. The work of W.D.L. is supported by the University of Maryland Center for String and Particle Theory.

A. 5D notation and conventions

Our 5D notation and conventions are very similar to those introduced in [28].

The 5D gamma-matrices $\Gamma_{\hat{m}} = (\Gamma_m, \Gamma_5)$, with $m = 0, 1, 2, 3$, defined by

$$\{\Gamma_{\hat{m}}\,,\,\Gamma_{\hat{n}}\} = -2\eta_{\hat{m}\hat{n}}\,\mathbf{1}\,, \qquad (\Gamma_{\hat{m}})^{\dagger} = \Gamma_0\,\Gamma_{\hat{m}}\,\Gamma_0 \tag{A.1}$$

are chosen in accordance with [27, 12]

$$(\Gamma_m)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} 0 & (\sigma_m)_{\alpha\dot{\beta}} \\ (\tilde{\sigma}_m)^{\dot{\alpha}\beta} & 0 \end{pmatrix}, \qquad (\Gamma_5)_{\hat{\alpha}}{}^{\hat{\beta}} = \begin{pmatrix} -\mathrm{i}\,\delta_{\alpha}{}^{\beta} & 0 \\ 0 & \mathrm{i}\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}, \tag{A.2}$$

such that $\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5 = \mathbf{1}$. The charge conjugation matrix, $C = (\varepsilon^{\hat{\alpha}\hat{\beta}})$, and its inverse, $C^{-1} = C^{\dagger} = (\varepsilon_{\hat{\alpha}\hat{\beta}})$ are defined by

$$C\,\Gamma_{\hat{m}}\,C^{-1} = (\Gamma_{\hat{m}})^{\mathrm{T}}\,, \qquad \varepsilon^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad \varepsilon_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & -\varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{A.3}$$

The antisymmetric matrices $\varepsilon^{\hat{\alpha}\hat{\beta}}$ and $\varepsilon_{\hat{\alpha}\hat{\beta}}$ are used to raise and lower the four-component spinor indices.

A Dirac spinor, $\Psi = (\Psi_{\hat\alpha})$, and its Dirac conjugate, $\Psi = (\bar\Psi^{\hat\alpha}) = \Psi^\dagger\,\Gamma_0$, look like

$$\Psi_{\hat\alpha} = \begin{pmatrix} \psi_\alpha \\ \bar\phi^{\dot\alpha} \end{pmatrix}\,, \qquad \bar\Psi^{\hat\alpha} = (\phi^\alpha\,,\,\bar\psi_{\dot\alpha})\;. \tag{A.4}$$

One can now combine $\bar\Psi^{\hat\alpha} = (\phi^\alpha, \bar\psi_{\dot\alpha})$ and $\Psi^{\hat\alpha} = \varepsilon^{\hat\alpha\hat\beta}\Psi_{\hat\beta} = (\psi^\alpha, -\bar\phi_{\dot\alpha})$ into a SU(2) doublet,

$$\Psi_i^{\hat\alpha} = (\Psi_i^\alpha, -\bar\Psi_{\dot\alpha i})\,, \qquad (\Psi_i^\alpha)^* = \bar\Psi^{\dot\alpha i}\,, \qquad i = \underline{1}, \underline{2}\,, \tag{A.5}$$

with $\Psi_{\underline{1}}^\alpha = \phi^\alpha$ and $\Psi_{\underline{2}}^\alpha = \psi^\alpha$. It is understood that the SU(2) indices are raised and lowered by ε^{ij} and ε_{ij}, $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$, in the standard fashion: $\Psi^{\hat\alpha i} = \varepsilon^{ij}\,\Psi_j^{\hat\alpha}$. The Dirac spinor $\Psi^i = (\Psi_{\hat\alpha}^i)$ satisfies the pseudo-Majorana condition $\bar\Psi_i{}^{\mathrm{T}} = C\,\Psi_i$. This will be concisely represented as

$$(\Psi_{\hat\alpha}^i)^* = \Psi_i^{\hat\alpha}\;. \tag{A.6}$$

With the definition $\Sigma_{\hat m\hat n} = -\Sigma_{\hat n\hat m} = -\frac{1}{4}[\Gamma_{\hat m}, \Gamma_{\hat n}]$, the matrices $\{\mathbf{1}, \Gamma_{\hat m}, \Sigma_{\hat m\hat n}\}$ form a basis in the space of 4×4 matrices. The matrices $\varepsilon_{\hat\alpha\hat\beta}$ and $(\Gamma_{\hat m})_{\hat\alpha\hat\beta}$ are antisymmetric, $\varepsilon^{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta} = 0$, while the matrices $(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}$ are symmetric. Given a 5-vector $V^{\hat m}$ and an antisymmetric tensor $F^{\hat m\hat n} = -F^{\hat n\hat m}$, we can equivalently represent them as the bi-spinors $V = V^{\hat m}\,\Gamma_{\hat m}$ and $F = \frac{1}{2}F^{\hat m\hat n}\,\Sigma_{\hat m\hat n}$ with the following symmetry properties

$$V_{\hat\alpha\hat\beta} = -V_{\hat\beta\hat\alpha}\,, \quad \varepsilon^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta} = 0\,, \qquad F_{\hat\alpha\hat\beta} = F_{\hat\beta\hat\alpha}\;. \tag{A.7}$$

The two equivalent descriptions $V_{\hat m} \leftrightarrow V_{\hat\alpha\hat\beta}$ and and $F_{\hat m\hat n} \leftrightarrow F_{\hat\alpha\hat\beta}$ are explicitly described as follows:

$$\begin{aligned} V_{\hat\alpha\hat\beta} &= V^{\hat m}\,(\Gamma_{\hat m})_{\hat\alpha\hat\beta}\,, \qquad & V_{\hat m} &= -\frac{1}{4}\,(\Gamma_{\hat m})^{\hat\alpha\hat\beta}\,V_{\hat\alpha\hat\beta}\,, \\ F_{\hat\alpha\hat\beta} &= \frac{1}{2}F^{\hat m\hat n}(\Sigma_{\hat m\hat n})_{\hat\alpha\hat\beta}\,, \qquad & F_{\hat m\hat n} &= (\Sigma_{\hat m\hat n})^{\hat\alpha\hat\beta}\,F_{\hat\alpha\hat\beta}\;. \end{aligned} \tag{A.8}$$

These results can be easily checked using the identities (see e.g. [26]):

$$\begin{aligned} \varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} &= \varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta} + \varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\delta\hat\beta} + \varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma}\,, \\ \varepsilon_{\hat\alpha\hat\gamma}\,\varepsilon_{\hat\beta\hat\delta} - \varepsilon_{\hat\alpha\hat\delta}\,\varepsilon_{\hat\beta\hat\gamma} &= -\frac{1}{2}\,(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} + \frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta}\,, \end{aligned} \tag{A.9}$$

and therefore

$$\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta} = \frac{1}{2}\,(\Gamma^{\hat m})_{\hat\alpha\hat\beta}\,(\Gamma_{\hat m})_{\hat\gamma\hat\delta} + \frac{1}{2}\,\varepsilon_{\hat\alpha\hat\beta}\,\varepsilon_{\hat\gamma\hat\delta}\,, \tag{A.10}$$

with $\varepsilon_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ the completely antisymmetric fourth-rank tensor.

Complex conjugation gives

$$(\varepsilon_{\hat\alpha\hat\beta})^* = -\varepsilon^{\hat\alpha\hat\beta}\,, \qquad (V_{\hat\alpha\hat\beta})^* = V^{\hat\alpha\hat\beta}\,, \qquad (F_{\hat\alpha\hat\beta})^* = F^{\hat\alpha\hat\beta}\,, \tag{A.11}$$

provided $V^{\hat m}$ and $F^{\hat m\hat n}$ are real.

The conventional 5D simple superspace $\mathbb{R}^{5|8}$ is parametrized by coordinates $z^{\hat{A}} = (x^{\hat{a}}, \theta_i^{\hat{\alpha}})$. Then, a hypersurface $x^5 = \text{const}$ in $\mathbb{R}^{5|8}$ can be identified with the 4D, $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$ parametrized by

$$z^A = (x^a, \theta_i^\alpha, \bar{\theta}^i_{\dot{\alpha}}), \qquad (\theta_i^\alpha)^* = \bar{\theta}^{\dot{\alpha} i} . \tag{A.12}$$

The Grassmann coordinates of $\mathbb{R}^{5|8}$ and $\mathbb{R}^{4|8}$ are related to each other as follows:

$$\theta_i^{\hat{\alpha}} = (\theta_i^\alpha, -\bar{\theta}_{\dot{\alpha} i}), \qquad \theta^i_{\hat{\alpha}} = \begin{pmatrix} \theta^i_\alpha \\ \bar{\theta}^{\dot{\alpha} i} \end{pmatrix} . \tag{A.13}$$

Interpreting x^5 as a central charge variable, one can view $\mathbb{R}^{5|8}$ as a 4D, $\mathcal{N} = 2$ central charge superspace (see below).

The flat covariant derivatives $D_{\hat{A}} = (\partial_{\hat{a}}, D^i_{\hat{\alpha}})$ obey the algebra

$$\{D^i_{\hat{\alpha}}\,,\, D^j_{\hat{\beta}}\} = -2\mathrm{i}\, \varepsilon^{ij} \Big((\Gamma^{\hat{c}})_{\hat{\alpha}\hat{\beta}}\, \partial_{\hat{c}} + \varepsilon_{\hat{\alpha}\hat{\beta}}\, \Delta \Big), \qquad [D^i_{\hat{\alpha}}\,,\, \partial_{\hat{b}}] = [D^i_{\hat{\alpha}}\,,\, \Delta] = 0\,, \tag{A.14}$$

or equivalently

$$[D_{\hat{A}}\,,\, D_{\hat{B}}\} = T_{\hat{A}\hat{B}}{}^{\hat{C}}\, D_{\hat{C}} + C_{\hat{A}\hat{B}}\, \Delta\,, \tag{A.15}$$

with Δ the central charge. The spinor covariant derivatives are

$$D^i_{\hat{\alpha}} = \frac{\partial}{\partial \theta_i^{\hat{\alpha}}} - \mathrm{i}\, (\Gamma^{\hat{b}})_{\hat{\alpha}\hat{\beta}}\, \theta^{\hat{\beta} i}\, \partial_{\hat{b}} - \mathrm{i}\, \theta^i_{\hat{\alpha}}\, \Delta\,. \tag{A.16}$$

One can relate the operators

$$D^i \equiv (D^i_{\hat{\alpha}}) = \begin{pmatrix} D^i_\alpha \\ \bar{D}^{\dot{\alpha} i} \end{pmatrix}, \qquad \bar{D}_i \equiv (D^{\hat{\alpha}}_i) = (D^\alpha_i\,,\, -\bar{D}_{\dot{\alpha} i}) \tag{A.17}$$

to the 4D, $\mathcal{N} = 2$ covariant derivatives $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot{\alpha}}_i)$ where [20, 56]

$$\begin{aligned} D^i_\alpha &= \frac{\partial}{\partial \theta^\alpha_i} + \mathrm{i}\, (\sigma^b)_{\alpha\dot{\beta}}\, \bar{\theta}^{\dot{\beta} i}\, \partial_b - \mathrm{i}\, \theta^i_\alpha\, (\Delta + \mathrm{i}\, \partial_5)\,, \\ \bar{D}_{\dot{\alpha} i} &= -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha} i}} - \mathrm{i}\, \theta^\beta_i (\sigma^b)_{\beta\dot{\alpha}}\, \partial_b - \mathrm{i}\, \bar{\theta}_{\dot{\alpha} i}\, (\Delta - \mathrm{i}\, \partial_5)\,. \end{aligned} \tag{A.18}$$

These operators obey the anti-commutation relations

$$\begin{aligned} \{D^i_\alpha\,,\, D^j_\beta\} &= -2\mathrm{i}\, \varepsilon^{ij}\, \varepsilon_{\alpha\beta}\, (\Delta + \mathrm{i}\, \partial_5)\,, \qquad \{\bar{D}_{\dot{\alpha} i}\,,\, \bar{D}_{\dot{\beta} j}\} = 2\mathrm{i}\, \varepsilon_{ij}\, \varepsilon_{\dot{\alpha}\dot{\beta}}\, (\Delta - \mathrm{i}\, \partial_5)\,, \\ \{D^i_\alpha\,,\, \bar{D}_{\dot{\beta} j}\} &= -2\mathrm{i}\, \delta^i_j\, (\sigma^c)_{\alpha\dot{\beta}}\, \partial_c\,, \end{aligned} \tag{A.19}$$

which correspond to the 4D, $\mathcal{N} = 2$ supersymmetry algebra with a complex central charge (see also [38]).

In terms of the operators (A.17), the operation of complex conjugation acts as follows

$$(D^i\, F)^\dagger\, \Gamma_0 = -(-1)^{\epsilon(F)}\, \bar{D}_i\, F^*\,, \tag{A.20}$$

with F an arbitrary superfield and $\epsilon(F)$ its Grassmann parity. This can be concisely represented as

$$(D^i_{\hat{\alpha}}\, F)^* = -(-1)^{\epsilon(F)}\, D^{\hat{\alpha}}_i\, F^*\,. \tag{A.21}$$

B. Tensor fields on the two-sphere

In this appendix we recall, following [25], the well-known one-to-one correspondence between smooth tensor fields on $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ and smooth scalar functions over SU(2) with definite U(1) charges. The two-sphere is obtained from SU(2) by factorization with respect to the equivalence relation

$$u^{+i} \sim \mathrm{e}^{\mathrm{i}\varphi} u^{+i} \qquad \varphi \in \mathbb{R} \,. \tag{B.1}$$

We start by introducing two open charts forming an atlas on SU(2) which, upon identificationon (B.1), leads to a useful atlas on S^2. The north patch is defined by

$$u^{+\underline{1}} \neq 0\,, \tag{B.2}$$

and here we can represent

$$\begin{aligned} u^{+i} &= u^{+\underline{1}}\, w^i\,, \qquad & w^i &= (1, u^{+\underline{2}}/u^{+\underline{1}}) = (1, w)\,, \\ u^-_i &= \overline{u^{+\underline{1}}}\, \bar{w}_i\,, \qquad & \bar{w}_i &= (1, \bar{w})\,, \qquad |u^{+\underline{1}}|^2 = (1 + w\bar{w})^{-1} \,. \end{aligned} \tag{B.3}$$

The south patch is defined by

$$u^{+\underline{2}} \neq 0\,, \tag{B.4}$$

and here we have

$$\begin{aligned} u^{+i} &= u^{+\underline{2}}\, y^i\,, \qquad & y^i &= (u^{+\underline{1}}/u^{+\underline{2}}, 1) = (y, 1)\,, \\ u^-_i &= \overline{u^{+\underline{2}}}\, \bar{y}_i\,, \qquad & \bar{y}_i &= (\bar{y}, 1)\,, \qquad |u^{+\underline{2}}|^2 = (1 + y\bar{y})^{-1} \,. \end{aligned} \tag{B.5}$$

In the overlap of the two charts we have

$$u^{+i} = \frac{\mathrm{e}^{\mathrm{i}\alpha}}{\sqrt{(1 + w\bar{w})}}\, w^i = \frac{\mathrm{e}^{\mathrm{i}\beta}}{\sqrt{(1 + y\bar{y})}}\, y^i\,, \tag{B.6}$$

where

$$y = \frac{1}{w}\,, \qquad \mathrm{e}^{\mathrm{i}\beta} = \sqrt{\frac{w}{\bar{w}}}\, \mathrm{e}^{\mathrm{i}\alpha} \,. \tag{B.7}$$

The variables w and y are seen to be local complex coordinates on S^2 considered as the Riemann sphere, $S^2 = \mathbb{C} \cup \{\infty\}$; the north chart $U_{\mathrm{N}} = \mathbb{C}$ is parametrized by w and the south patch $U_{\mathrm{S}} = \mathbb{C}^* \cup \{\infty\}$ is parametrized by y.

Along with w^i and $\bar{w}_i$, we often use their counterparts with lower (upper) indices

$$w_i = \varepsilon_{ij} w^j = (-w, 1)\,, \qquad \bar{w}^i = \varepsilon^{ij} \bar{w}_j = (\bar{w}, -1)\,, \qquad \overline{w_i} = -\bar{w}^i\,, \tag{B.8}$$

and similar for y_i and $\bar{y}^i$.

Let $\Psi^{(p)}(u)$ be a smooth function on SU(2) with U(1)-charge p chosen, for definiteness, to be non-negative, $p \geq 0$. Such a function possesses a convergent Fourier series of the form

$$\Psi^{(p)}(u) = \sum_{n=0}^{\infty} \Psi^{(i_1 \cdots i_{n+p} j_1 \cdots j_n)} u^+_{i_1} \cdots u^+_{i_{n+p}} u^-_{j_1} \cdots u^-_{j_n}\,, \qquad p \geq 0 \,. \tag{B.9}$$

In the north patch we can write

$$
\begin{aligned}
\Psi^{(p)}(u) &= (u^{+\underline{1}})^p \; \Psi^{(p)}_{\rm N}(w,\bar{w})\,, \\
\Psi^{(p)}_{\rm N}(w,\bar{w}) &= \sum_{n=0}^{\infty} \Psi^{(i_1\cdots i_{n+p} j_1\cdots j_n)} \; \frac{w_{i_1}\cdots w_{i_{n+p}} \bar{w}_{j_1}\cdots \bar{w}_{j_n}}{(1+w\bar{w})^n}\;.
\end{aligned}
\tag{B.10}
$$

In the south patch we have

$$
\begin{aligned}
\Psi^{(p)}(u) &= (u^{+\underline{2}})^p \; \Psi^{(p)}_{\rm S}(y,\bar{y})\,, \\
\Psi^{(p)}_{\rm S}(y,\bar{y}) &= \sum_{n=0}^{\infty} \Psi^{(i_1\cdots i_{n+p} j_1\cdots j_n)} \; \frac{y_{i_1}\cdots y_{i_{n+p}} \bar{y}_{j_1}\cdots \bar{y}_{j_n}}{(1+y\bar{y})^n}\;.
\end{aligned}
\tag{B.11}
$$

Finally, in the overlap of the two charts $\Psi^{(p)}_{\rm N}$ and $\Psi^{(p)}_{\rm S}$ are simply related to each other

$$
\Psi^{(p)}_{\rm S}(y,\bar{y}) = \frac{1}{w^p}\; \Psi^{(p)}_{\rm N}(w,\bar{w})\,, \tag{B.12}
$$

If we redefine

$$
\hat{\Psi}^{(p)}_{\rm N}(w,\bar{w}) = {\rm e}^{{\rm i}p\pi/4}\; \Psi^{(p)}_{\rm N}(w,\bar{w})\,, \qquad \check{\Psi}^{(p)}_{\rm S}(y,\bar{y}) = {\rm e}^{-{\rm i}p\pi/4}\; \Psi^{(p)}_{\rm S}(y,\bar{y})\,,
$$

the above relation takes the form

$$
\check{\Psi}^{(p)}_{\rm S}(y,\bar{y}) = \left(\frac{\partial y}{\partial w}\right)^{p/2} \; \hat{\Psi}^{(p)}_{\rm N}(w,\bar{w}) \tag{B.13}
$$

and thus defines a smooth tensor field on S^2.

C. Projective superspace action

In this appendix we briefly demonstrate, following [25], how to derive the projective superspace action (6.14) from the harmonic superspace action (3.2). More details can be found in [25].

Consider an arbitrary projective superfield $\phi(z,w)$, eq. (6.10), which is allowed to be singular only at $w=0$ and $w=\infty$ (i.e. $\phi(z,w)$ is holomorphic on the doubly punctured sphere $S^2\backslash\{N\cup S\}$). It is possible to promote $\phi(z,w)$ to a smooth analytic superfield over S^2 by smearing (regularizing) its singularities with functions used in the construction of the partition of unity in differential geometry.

Consider a smooth cut-off function $F_{R,\epsilon}(x)$ sketched in figure 3.

This function extrapolates smoothly from unit magnitude to zero in a small region between R, with is assumed to be large number, and $R+\epsilon$ where ϵ is small. The derivative of this function localizes whatever it multiplies to this region and is normalized so that in the limit

$$
\lim_{\epsilon\longrightarrow 0} F'_{R,\epsilon}(x) = -\delta(x-R) \tag{C.1}
$$

JHEP02(2006)038

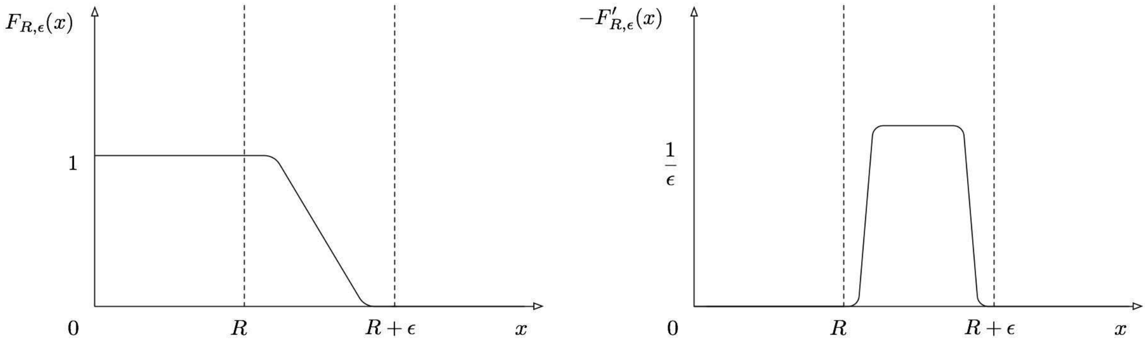

Figure 3: The function $F_{R,\epsilon}(x)$ smoothy interpolates from 1 to 0 in the region of width ϵ starting at R. It's derivative is a bump function with support $[R, R+\epsilon]$ and unit area.

as a distribution. Now, we can regularize the projective superfield $\phi(z,w)$ as follows:

$$\phi(z,w) \quad \longrightarrow \quad \phi_{R,\epsilon}(z,w,\bar{w}) = F_{R,\epsilon}(|w|^{-1})\,\phi(z,w)\,F_{R,\epsilon}(|w|)\,, \tag{C.2}$$

and the result is a a smooth neutral analytic superfield over the harmonic superspsace. If $\phi(z,w)$ is regular at $w=0$ or $w=\infty$, then the factor $F_{R,\epsilon}(|w|^{-1})$ or $F_{R,\epsilon}(|w|)$ on the right of (C.2) can be removed.

The above procedure can also be used to generate charged analytic superfields. For instance, if $\Lambda(z,w)$ is a real projective superfield, $\breve{\Lambda} = \Lambda$, then the following superfields

$$L^{++}_{R,\epsilon}(z,u) = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\,F_{R,\epsilon}(|w|^{-1})\,L(z,w)\,F_{R,\epsilon}(|w|) \equiv \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\,L_{R,\epsilon}(z,w,\bar{w})\,, \tag{C.3}$$

$$L^{(+4)}_{R,\epsilon}(z,u) = (u^{+\underline{1}}u^{+\underline{2}})^2 F_{R,\epsilon}(|w|^{-1})\,L(z,w)\,F_{R,\epsilon}(|w|) \equiv (u^{+\underline{1}}u^{+\underline{2}})^2 L_{R,\epsilon}(z,w,\bar{w}) \tag{C.4}$$

are real analytic superfields of charge $+2$ and $+4$, respectively. One can use $L^{(+4)}_{R,\epsilon}(z,u)$ in the role of Lagrangian in (3.2). In the final stages we will remove the regulator by taking first $\epsilon \to 0$ and then $R \to \infty$.

As is seen from (3.2) and (3.3), the analytic action involves a square of $(\hat{D}^-)^2$, and therefore we sould express the operators $(\hat{D}^-)^2$ in local coordinates. What actually we need here is this operator acting on analytic or projective superfields Φ such that $\nabla_\alpha(w)\Phi = \bar{\nabla}^{\dot{\alpha}}(w)\Phi = 0$, with operators ∇_α and $\bar{\nabla}^{\dot{\alpha}}(w)$ defined in (6.12). The analyticity allows us to move all $\mathcal{D}^{\underline{2}}_\alpha$ and $\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}}$ derivatives onto Φ and rewrite them in terms of $D^{\underline{1}}_\alpha$ and $\bar{D}^{\dot{\alpha}}_{\underline{1}}$. When this is done, we find in local coordinates for an analytic Φ

$$(\hat{D}^-)^2\Phi = -4\,(\overline{u^{+\underline{1}}})^2\,\frac{(1+\bar{w}w)^2}{w}\,\mathcal{P}(w)\Phi\,, \tag{C.5}$$

where we have defined the projective differential operator

$$\mathcal{P}(w) = \frac{1}{4w}\,(\bar{D}_{\underline{1}})^2 + \partial_5 - \frac{w}{4}\,(D^{\underline{1}})^2\,. \tag{C.6}$$

It is worth pointing out that eq. (C.5) also holds in the presence of a non-vanishing central charge Δ. Using the analyticity of Φ again, it is easy to show that

$$(\hat{D}^-)^4\Phi = (\overline{u^{+\underline{1}}})^4\,\frac{(1+\bar{w}w)^4}{w^2}D^4\Phi \; + \; \text{total derivatives}\,, \tag{C.7}$$

with the D^4 operator defined by (6.16). The latter operator determines the projective superspace measure, see eq. (6.15). Finally making use of the identity

$$\mathrm{d}u = \frac{\mathrm{d}^2 w}{\pi(1+w\bar{w})^2}\,, \tag{C.8}$$

one obtains (note $|u^{+\underline{1}}|^2 = (1+w\bar{w})^{-1}$)

$$\int \mathrm{d}\zeta^{(-4)}\, L^{(+4)}_{R,\epsilon}(z,u) = \frac{1}{\pi}\int \mathrm{d}^5 x \int \frac{\mathrm{d}^2 w}{(1+w\bar{w})^2}\, D^4 L_{R,\epsilon}(z,w,\bar{w})\Big\| \,. \tag{C.9}$$

Representing here

$$\frac{1}{(1+w\bar{w})^2} = -\frac{1}{w}\,\partial_{\bar{w}}\,\frac{1}{(1+w\bar{w})}$$

and integrating by parts, one can then show

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\int \mathrm{d}\zeta^{(-4)}\, L^{(+4)}_{R,\epsilon}(z,u) = \frac{1}{2\pi \mathrm{i}}\int \mathrm{d}^5 x \oint \frac{\mathrm{d}w}{w}\, D^4 L(z,w)\Big\| \,. \tag{C.10}$$

This is exactly the projective action.

The formalism developed in this appendix can be applied to obtain a nice representation for the supersymmetric action (3.12) which is equivalent to

$$S = \frac{\mathrm{i}}{4}\int \mathrm{d}^5 x \int \mathrm{d}u\,(\hat{D}^-)^2 L^{++}\Big\|\,, \qquad D^+_{\hat{\alpha}} L^{++} = 0\,, \quad D^{++}L^{++} = 0\,. \tag{C.11}$$

Representing $L^{++} = \mathrm{i}u^{+\underline{1}}u^{+\underline{2}}\, L(z,w)$ and using eq. (C.5), we obtain

$$S = \int \mathrm{d}^5 x \int \mathrm{d}u\, \mathcal{P}(w)\, L(z,w)\Big\| \,. \tag{C.12}$$

Finally, making use of (7.7) gives

$$S = \frac{1}{8\pi \mathrm{i}}\int \mathrm{d}^5 x \oint \frac{\mathrm{d}w}{w}\left[\frac{1}{w}\bar{D}^2 - wD^2\right] L(z,w)\Big\| \,. \tag{C.13}$$

As an example of the usefulness of such a form, we can consider the super Yang-Mills Lagrangian (3.18). A trivial contour integration in (C.13) then immediately reproduces the action for this theory in reduced superspace (5.11).

References

[1] I. Antoniadis, *A possible new dimension at a few TeV*, *Phys. Lett.* **B 246** (1990) 377.

[2] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142];
P. Hořava, *Gluino condensation in strongly coupled heterotic string theory*, *Phys. Rev.* **D 54** (1996) 7561 [hep-th/9608019].

[3] G.R. Dvali and M.A. Shifman, *Dynamical compactification as a mechanism of spontaneous supersymmetry breaking*, *Nucl. Phys.* **B 504** (1997) 127 [hep-th/9611213].

JHEP02(2006)038

[4] L. Randall and R. Sundrum, *Out of this world supersymmetry breaking*, *Nucl. Phys.* **B 557** (1999) 79 [hep-th/9810155].

[5] G.F. Giudice, M.A. Luty, H. Murayama and R. Rattazzi, *Gaugino mass without singlets*, *JHEP* **12** (1998) 027 [hep-ph/9810442];
Z. Chacko and M.A. Luty, *Radion mediated supersymmetry breaking*, *JHEP* **05** (2001) 067 [hep-ph/0008103].

[6] T. Gregoire, R. Rattazzi and C.A. Scrucca, *D-type supersymmetry breaking and brane-to-brane gravity mediation*, hep-ph/0505126.

[7] R. Rattazzi, C.A. Scrucca and A. Strumia, *Brane to brane gravity mediation of supersymmetry breaking*, *Nucl. Phys.* **B 674** (2003) 171 [hep-th/0305184].

[8] W.D. Linch III, M.A. Luty and J. Phillips, *Five dimensional supergravity in $N = 1$ superspace*, *Phys. Rev.* **D 68** (2003) 025008 [hep-th/0209060].

[9] I.L. Buchbinder, S.J. Gates, H.S.J. Goh, W.D. Linch III, M.A. Luty, S.P. Ng and J. Phillips, *Supergravity loop contributions to brane world supersymmetry breaking*, *Phys. Rev.* **D 70** (2004) 025008 [hep-th/0305169].

[10] T. Gregoire, R. Rattazzi, C.A. Scrucca, A. Strumia and E. Trincherini, *Gravitational quantum corrections in warped supersymmetric brane worlds*, *Nucl. Phys.* **B 720** (2005) 3 [hep-th/0411216].

[11] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [hep-th/0108200].

[12] I.L. Buchbinder and S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity or a walk through superspace*, IOP, Bristol 1998.

[13] P. Breitenlohner and A. Kabelschacht, *The auxiliary fields of $N = 2$ extended supergravity in 5 and 6 space-time dimensions*, *Nucl. Phys.* **B 148** (1979) 96.

[14] P.S. Howe, *Off-shell $N = 2$ and $N = 4$ supergravity in five-dimensions*, in *Quantum structure of space and time*, M.J. Duff and C.J. Isham eds., Cambridge University Press, Cambridge 1982.

[15] P.S. Howe and U. Lindström, *The supercurrent in five-dimensions*, *Phys. Lett.* **B 103** (1981) 422.

[16] M. Zucker, *Minimal off-shell supergravity in five dimensions*, *Nucl. Phys.* **B 570** (2000) 267 [hep-th/9907082]; *Gauged $N = 2$ off-shell supergravity in five dimensions*, *JHEP* **08** (2000) 016 [hep-th/9909144]; *Supersymmetric brane world scenarios from off-shell supergravity*, *Phys. Rev.* **D 64** (2001) 024024 [hep-th/0009083]; *Off-shell supergravity in five-dimensions and supersymmetric brane world scenarios*, *Fortschr. Phys.* **51** (2003) 899.

[17] T. Kugo and K. Ohashi, *Off-shell $D = 5$ supergravity coupled to matter–Yang-Mills system*, *Prog. Theor. Phys.* **105** (2001) 323 [hep-ph/0010288];
T. Fujita and K. Ohashi, *Superconformal tensor calculus in five dimensions*, *Prog. Theor. Phys.* **106** (2001) 221 [hep-th/0104130];
T. Fujita, T. Kugo and K. Ohashi, *Off-shell formulation of supergravity on orbifold*, *Prog. Theor. Phys.* **106** (2001) 671 [hep-th/0106051];
T. Kugo and K. Ohashi, *Superconformal tensor calculus on orbifold in 5d*, *Prog. Theor. Phys.* **108** (2002) 203 [hep-th/0203276].

[18] E. Bergshoeff et al., *Superconformal $N = 2$, $D = 5$ matter with and without actions*, *JHEP* **10** (2002) 045 [hep-th/0205230];
E. Bergshoeff et al., *$N = 2$ supergravity in five dimensions revisited*, *Class. and Quant. Grav.* **21** (2004) 3015 [hep-th/0403045].

[19] F. Paccetti Correia, M.G. Schmidt and Z. Tavartkiladze, *Superfield approach to 5d conformal sugra and the radion*, *Nucl. Phys.* **B 709** (2005) 141 [hep-th/0408138];
F.P. Correia, M.G. Schmidt and Z. Tavartkiladze, *The radion in 5d conformal sugra: superspace action*, *Fortschr. Phys.* **53** (2005) 592 [hep-th/0501209].

[20] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469.

[21] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge 2001.

[22] E. Sokatchev, *Off-shell six-dimensional supergravity in harmonic superspace*, *Class. and Quant. Grav.* **5** (1988) 1459.

[23] A. Karlhede, U. Lindström and M. Roček, *Selfinteracting tensor multiplets in $N = 2$ superspace*, *Phys. Lett.* **B 147** (1984) 297.

[24] U. Lindström and M. Roček, *New hyperkahler metrics and new supermultiplets*, *Commun. Math. Phys.* **115** (1988) 21;
U. Lindström and M. Roček, *$N = 2$ super Yang-Mills theory in projective superspace*, *Commun. Math. Phys.* **128** (1990) 191;
F. Gonzalez-Rey, M. Roček, S. Wiles, U. Lindström and R. von Unge, *Feynman rules in $N = 2$ projective superspace, I. Massless hypermultiplets*, *Nucl. Phys.* **B 516** (1998) 426 [hep-th/9710250].

[25] S.M. Kuzenko, *Projective superspace as a double-punctured harmonic superspace*, *Int. J. Mod. Phys.* **A 14** (1999) 1737 [hep-th/9806147].

[26] B. Zupnik, *Harmonic superpotentials and symmetries in gauge theories with eight supercharges*, *Nucl. Phys.* **B 554** (1999) 365 [hep-th/9902038], erratum *ibid.* **B 644** (2002) 405.

[27] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[28] J. Bagger and D.V. Belyaev, *Supersymmetric branes with (almost) arbitrary tensions*, *Phys. Rev.* **D 67** (2003) 025004 [hep-th/0206024].

[29] A.A. Rosly, *Super Yang-Mills constraints as integrability conditions*, in *Group theoretical methods in physics*, M.A. Markov ed., Nauka, Moscow 1983; *Gauge fields in superspace and twistors*, *Class. and Quant. Grav.* **2** (1985) 693;
A.A. Rosly and A.S. Schwarz, *Supersymmetry in a space with auxiliary dimensions*, *Commun. Math. Phys.* **105** (1986) 645.

[30] B.M. Zupnik, *The action of the supersymmetric $N = 2$ gauge theory in harmonic superspace*, *Phys. Lett.* **B 183** (1987) 175.

[31] P. Fayet, *Fermi-Bose hypersymmetry*, *Nucl. Phys.* **B 113** (1976) 135.

[32] M.F. Sohnius, *Supersymmetry and central charges*, *Nucl. Phys.* **B 138** (1978) 109.

[33] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko and B.A. Ovrut, *The background field method for $N = 2$ super Yang-Mills theories in harmonic superspace*, *Phys. Lett.* **B 417** (1998) 61 [hep-th/9704214];
I.L. Buchbinder, S.M. Kuzenko and B.A. Ovrut, *On the $D = 4$, $N = 2$ non-renormalization theorem*, *Phys. Lett.* **B 433** (1998) 335 [hep-th/9710142].

[34] I.L. Buchbinder, E.I. Buchbinder, E.A. Ivanov, S.M. Kuzenko and B.A. Ovrut, *Effective action of the $N = 2$ Maxwell multiplet in harmonic superspace*, *Phys. Lett.* **B 412** (1997) 309 [hep-th/9703147];
I.L. Buchbinder and S.M. Kuzenko, *The off-shell massive hypermultiplets revisited*, *Class. and Quant. Grav.* **14** (1997) L157 [hep-th/9704002];
N. Dragon and S.M. Kuzenko, *The Higgs mechanism in $N = 2$ superspace*, *Nucl. Phys.* **B 508** (1997) 229 [hep-th/9705027];
E.A. Ivanov, S.V. Ketov and B.M. Zupnik, *Induced hypermultiplet self-interactions in $N = 2$ gauge theories*, *Nucl. Phys.* **B 509** (1998) 53 [hep-th/9706078].

[35] N. Ohta, H. Sugata and H. Yamaguchi, *$N = 2$ harmonic superspace with central charges and its application to selfinteracting massive hypermultiplets*, *Ann. Phys. (NY)* **172** (1986) 26.

[36] M. Henningson, *Extended superspace, higher derivatives and $\mathrm{SL}(2, \mathbb{Z})$ duality*, *Nucl. Phys.* **B 458** (1996) 445 [hep-th/9507135].

[37] N. Dragon, S.M. Kuzenko and U. Theis, *The vector-tensor multiplet in harmonic superspace*, *Eur. Phys. J.* **C 4** (1998) 717 [hep-th/9706169].

[38] N. Dragon, E. Ivanov, S. Kuzenko, E. Sokatchev and U. Theis, *$N = 2$ rigid supersymmetry with gauged central charge*, *Nucl. Phys.* **B 538** (1999) 411 [hep-th/9805152].

[39] S.M. Kuzenko and S. Theisen, *Correlation functions of conserved currents in $N = 2$ superconformal theory*, *Class. and Quant. Grav.* **17** (2000) 665 [hep-th/9907107].

[40] R. Grimm, M. Hasler and C. Herrmann, *The $N = 2$ vector-tensor multiplet, central charge superspace and Chern-Simons couplings*, *Int. J. Mod. Phys.* **A 13** (1998) 1805 [hep-th/9706108].

[41] I. Buchbinder, A. Hindawi and B.A. Ovrut, *A two-form formulation of the vector-tensor multiplet in central charge superspace*, *Phys. Lett.* **B 413** (1997) 79 [hep-th/9706216].

[42] M. Sohnius, K.S. Stelle and P.C. West, *Off mass shell formulation of extended supersymmetric gauge theories*, *Phys. Lett.* **B 92** (1980) 123;
M.F. Sohnius, K.S. Stelle and P.C. West, *Dimensional reduction by legendre transformation generates off-shell supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 173** (1980) 127.

[43] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative couplings of vector multiplets in $N = 2$ heterotic string vacua*, *Nucl. Phys.* **B 451** (1995) 53 [hep-th/9504006].

[44] N. Dragon and S.M. Kuzenko, *Self-interacting vector-tensor multiplet*, *Phys. Lett.* **B 420** (1998) 64 [hep-th/9709088];
E. Ivanov and E. Sokatchev, *On non-linear superfield versions of the vector-tensor multiplet*, *Phys. Lett.* **B 429** (1998) 35 [hep-th/9711038].

[45] P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia, *The vector-tensor supermultiplet with gauged central charge*, *Phys. Lett.* **B 373** (1996) 81 [hep-th/9512143].

[46] N. Arkani-Hamed, T. Gregoire and J. Wacker, *Higher dimensional supersymmetry in 4d superspace*, *JHEP* **03** (2002) 055 [hep-th/0101233];
D. Marti and A. Pomarol, *Supersymmetric theories with compact extra dimensions in $N = 1$ superfields*, *Phys. Rev.* **D 64** (2001) 105025 [hep-th/0106256];
A. Hebecker, *5d super Yang-Mills theory in 4-d superspace, superfield brane operators and applications to orbifold guts*, *Nucl. Phys.* **B 632** (2002) 101 [hep-ph/0112230].

[47] H.M. Farkas and I. Kra, *Riemann surfaces*, Springer, New-York 1992.

[48] S.J. Gates Jr., T. Hübsch and S.M. Kuzenko, *CNM models, holomorphic functions and projective superspace c-maps*, *Nucl. Phys.* **B 557** (1999) 443 [hep-th/9902211].

[49] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475.

[50] G. Sierra and P.K. Townsend, *An introduction to $N = 2$ rigid supersymmetry*, in *Supersymmetry and supergravity 1983*, B. Milewski ed., World Scientific, Singapore 1983;
B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su and A. Van Proeyen, *Gauge and matter fields coupled to $N = 2$ supergravity*, *Phys. Lett.* **B 134** (1984) 37;
S.J. Gates Jr., *Superspace formulation of new nonlinear sigma models*, *Nucl. Phys.* **B 238** (1984) 349.

[51] S.J. Gates Jr. and S.M. Kuzenko, *The CNM-hypermultiplet nexus*, *Nucl. Phys.* **B 543** (1999) 122 [hep-th/9810137]; *4d $N = 2$ supersymmetric off-shell sigma models on the cotangent bundles of Kähler manifolds*, *Fortschr. Phys.* **48** (2000) 115 [hep-th/9903013].

[52] B. Zumino, *Supersymmetry and Kähler manifolds*, *Phys. Lett.* **B 87** (1979) 203.

[53] S.M. Kuzenko, unpublished.

[54] D. Kaledin, *Hyperkähler structures on total spaces of holomorphic cotangent bundles*, in D. Kaledin and M. Verbitsky, *Hyperkähler manifolds*, International Press, Cambridge 1999 [alg-geom/9710026];
A canonical hyperkähler metric on the total space of a cotangent bundle, in *Quaternionic structures in mathematics and physics*, S. Marchiafava, P. Piccinni and M. Pontecorvo eds., World Scientific, Singapore 2001 [alg-geom/0011256];
B. Feix, *Hyperkähler metrics on cotangent bundles*, Cambridge PhD thesis, 1999, *J. Reine Angew. Math.* **532** (2001) 33.

[55] N. Seiberg, *Five dimensional SUSY field theories, non-trivial fixed points and string dynamics*, *Phys. Lett.* **B 388** (1996) 753 [hep-th/9608111];
D.R. Morrison and N. Seiberg, *Extremal transitions and five-dimensional supersymmetric field theories*, *Nucl. Phys.* **B 483** (1997) 229 [hep-th/9609070];
K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys.* **B 497** (1997) 56 [hep-th/9702198].

[56] N. Ohta, *Dipole mechanism of spontaneous breaking of $N = 2$ supersymmetry, 2. Reformulation and generalization in harmonic superspace*, *Phys. Rev.* **D 32** (1985) 1467.

[57] S.J. Gates Jr., S. Penati and G. Tartaglino-Mazzucchelli, *6d supersymmetry, projective superspace and 4d, $N = 1$ superfields*, hep-th/0508187.

[58] M. Roček, C. Vafa and S. Vandoren, *Hypermultiplets and topological strings*, hep-th/0512206.